The quasi-lattice of indiscernible elements

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Abstract

The literature on quantum logic emphasizes that the algebraic structures involved with orthodox quantum mechanics are non distributive. In this paper we develop a particular algebraic structure, the *quasi-lattice* $\mathfrak{I}$-lattice, which can be modeled by an algebraic structure built in quasi-set theory $\mathfrak{Q}$. This structure is non distributive and involve indiscernible elements. Thus we show that in taking into account indiscernibility as a primitive concept, the quasi-lattice that 'naturally' arises is non distributive.

Introduction

Ever since the pioneering paper by Birkhoff and von Neumann (1936), the so called 'quantum logic' became a wide field of logic and several distinct logical systems, including some associated with non-classical logics, such as paraconsistent logics, have grown up. For an updated review of the field, see (Dalla Chiara, Giuntini, Greechie 2004) and (Rédei 2007). According to Birkhoff and von Neumann, the main aspect of quantum logic, in distinction to 'classical logic', which has as a model a Boolean lattice, is that it is not distributive, namely, the distributive law

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1}$$

does not hold. The explanations vary, but put in a nutshell, we can say that this reflects the fact that in quantum theories there are observable that do not commute, that is, they cannot be measured in conjunction; thus, although the first member of (1) holds, the second one does not. For historical details, see (Rédei 2007).

A further claim related to orthodox quantum physics is that the basic elements would be, in certain situations, absolute indiscernible, say when they enter in superposition states (mainly in the case of bosons). Some forerunners of quantum physics, like Heisenberg, Born, and Schrödinger, spoke of the lost of individuality and (Schrödinger) spoke even that the notion of identity of quantum entities should be meaningless. For historical details, see (French & Krause 2006).

Usually, physicists express this 'lost of identity' by assuming certain symmetry conditions *after* having assumed that the basic objects of their consideration are individuated, say by their coordinates or by some individual description, typical of the usual formalism. Really, in order to express indiscernibility, we begin by labeling the considered entities, say by calling them 'particle 1', 'particle 2', etc., thus reasoning as they were distinct entities, and then we apply symmetry conditions to represent the relevant states of the join systems. Saying in brief, physicists consider as relevant just functions (or vectors) which are invariant by any permutation of the labels of the 'indiscernible objects', such as the symmetric and anti-symmetric vector/functions. In a series of works, we have followed the idea of pursuing indiscernibility 'right from the start', to use H. Post's words (Post 1973). In a more adequate formalism, the indiscernibility, as a fundamental concept in quantum physics, would not be made *a posteriori*, but would be taken as a primitive concept (for details see French & Krause 2006). It is a trivial consequence that it would be interesting to extend this discussion to the algebraic aspects of a 'quantum logic' for situations involving indiscernibility. The aim of this paper is to start this program.

Without revising the history of quantum logic here, we introduce the $\mathfrak{I}$-quasi-lattice of indiscernible elements within the framework of the quasi-set theory, a mathematical theory built to deal with indiscernible elements 'directly', as we shall see below, that is, without using the standard mathematical trick of naming the objects first and keeping their discernibility in the sequence by introducing symmetry conditions. In order to keep the paper self-contained, we begin by revising some basic concepts which are common to quantum algebras. Below, we recall some aspects of the quasi set theory $\mathfrak{Q}$ that serves as our framework and, finally, we present a particular model of the $\mathfrak{I}$-quasi-lattice, which grow up in a natural way from indiscernible elements.

1 On quantum algebras

In order to keep the paper self-contained, we begin by recalling some basic notions of lattice theory and then we present some algebraic concepts proper of quantum logic.

Definition 1.1 (Lattice)**.** A *lattice* is an algebraic structure $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ formed by a nonempty set L and by two binary operations $\wedge$ and $\vee$ on L such that, for all $x, y, z \in L$, the following laws hold:

L_1 $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ [associativity];
L_2 $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ [commutativity];
L_3 $(x \wedge y) \vee y = y$ and $(x \vee y) \wedge y = y$ [absorption].

Theorem 1.2. *In any lattice, the following laws hold:*

L_4 $x \wedge x = x$ *and* $x \vee x = x$ [idempotency];
L_5 $x \wedge y = x$ iff $x \vee y = y$ [ordering].

From L_5, we have a very natural way to define a partial ordering relation in a lattice $\mathfrak{L} = \langle L, \wedge, \vee\rangle$, namely:

Definition 1.3 (Order)**.** $x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y.$

Theorem 1.4. *The following properties are also valid for the operations* $\wedge$ *and* $\vee$*:*

L_6 $x \leq x \vee y$ *and* $y \leq x \vee y;$
L_7 $x \wedge y \leq x$ *and* $x \wedge y \leq y;$
L_8 $x \leq z$ *and* $y \leq z \Rightarrow x \vee y \leq z;$
L_9 $z \leq x$ *and* $z \leq y \Rightarrow z \leq x \wedge y;$
L_{10} $x \leq z$ *and* $y \leq w \Rightarrow x \vee y \leq z \vee w;$
L_{11} $x \leq z$ *and* $y \leq w \Rightarrow x \wedge y \leq z \wedge w.$

The proof of above properties can be seen in (Rasiowa & Sikorski 1968).

We have introduced a lattice as an algebraic structure, however a lattice can also be seen as an ordering structure $\mathfrak{L} = \langle L, \leq\rangle$, as it is easy to see.

Definition 1.5. A *partially ordered set* (poset) is a pair $\langle L, \leq\rangle$ such that L is a nonempty set and $\leq$ is a partial order on L, that is, the relation $\leq$ is reflexive, antisymmetric, and transitive on L.

Definition 1.6. Let $\langle L, \leq\rangle$ be a poset and $a, b \in L$. If there exists an element $c \in L$ such that:

(i) $a \leq c$ and $b \leq c$

(ii) $a \leq d$ and $b \leq d \Rightarrow c \leq d$,
then c is the *supremum* of $\{a, b\}$.

Definition 1.7. Let $(L, \leq)$ a poset and $a, b \in L$. If there exists an element $e \in L$ such that:
(i) $e \leq a$ and $e \leq b$
(ii) $f \leq a$ and $f \leq b \Rightarrow f \leq e$,
then e is the *infimum* of $\{a, b\}$.

In general, we denote the supremum of $\{a, b\}$ by $\sup\{a, b\}$ or $a \vee b$ and the infimum of $\{a, b\}$ by $\inf\{a, b\}$ or $a \wedge b$. The supremum of $\{a, b\}$ is also named the *least upper bound* of $\{a, b\}$ and the infimum of $\{a, b\}$ is called the *greatest lower bound* of $\{a, b\}$.

Let $\langle L, \leq\rangle$ be an ordered set such that for all $a, b \in L$ there exist the $\inf\{a, b\}$ and the $\sup\{a, b\}$. Then the algebraic structure determined by $\langle L, \wedge, \vee\rangle$ in which

$$x \vee y = \sup\{x, y\} \quad \text{and} \quad x \wedge y = \inf\{x, y\}$$

is a *lattice*, as it is easy to see.

We can easily to proof that the laws L_1 to L_{11} hold for the poset $\langle L, \leq\rangle$.

Lemma 1.8. *In any lattice* $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ *we have that*
L_{12} $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$;
L_{13} $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$.

Proof: Immediate consequence of L_6, L_7, and L_8. ■

Definition 1.9. The lattice $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ is *distributive* if the following distributive laws are valid for all $x, y, z \in L$:
L_{14} $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ and $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

These distributive laws are known as the distributivity at left-hand side and, due to the commutative property, the distributive laws at right-hand side are also valid. Besides, only one of the two distributive laws listed above would be sufficient to characterize the commutative property.

Definition 1.10. If a lattice $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ has the least element given by the ordering $\leq$, then this element is the *zero* of $\mathfrak{L}$ and is denoted by 0. On the other hand, if the lattice $\mathfrak{L}$ has the greatest element, given by the ordering $\leq$, then this element is the *unit* of $\mathfrak{L}$ and it is denoted by 1.

If $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ has the zero, 0, and the unity, 1, then:
L_{15} $x \wedge 0 = 0$ and $x \vee 0 = x$;
L_{16} $x \wedge 1 = x$ and $x \vee 1 = 1$.

Definition 1.11. Let $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ be a lattice with 0 and 1. Given $x \in L$, an element $y \in L$ is a *complement* of x in $\mathfrak{L}$ if $x \wedge y = 0$ and $x \vee y = 1$. A lattice $\mathfrak{L}$ is *complemented* when every element in L has a complement.

In general, a complement of x is denoted by $\sim x$.

Definition 1.12. When every element of $\mathcal{L} = \langle L, \wedge, \vee \rangle$ has exactly one complement, then the lattice $\mathcal{L}$ is *uniquely complemented*.

Lemma 1.13. *Let $\mathcal{L} = \langle L, \wedge, \vee \rangle$ be a distributive lattice with 0 and 1. If there exists a complement of x, then it is unique.*

Proof: If y and z are two complements of x, then $x \wedge y = 0$, $x \vee y = 1$, $x \wedge z = 0$, and $x \vee z = 1$. From that $z = 0 \vee z = (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z) = 1 \wedge (y \vee z) = y \vee z$, that is, $y \leq z$. Analogously, $z \leq y$ and, hence, $z = y$. ■

Definition 1.14. A *Boolean algebra* $\mathcal{B}$ is a distributive and complemented lattice.

Definition 1.15. Let $\mathcal{L} = \langle L, \wedge, \vee \rangle$ be a lattice with 0 and 1. Given $x \in L$, if there exists the element $-x = \max\{y \in L : x \wedge y = 0\}$ in L, then x is *pseudo-complemented* and $-x$ is its *pseudo-complement*. The lattice $\mathcal{L}$ is *pseudo-complemented* when every element $x \in L$ has a pseudo-complement $-x$.

Definition 1.16. A lattice $\mathcal{L} = \langle L, \wedge, \vee \rangle$ is *orthocomplemented* if it is complemented and for all $a, b \in L$,

L_{17} $\sim\sim a = a$;

L_{18} $a \leq b \Rightarrow \sim b \leq \sim a$.

Theorem 1.17. *If $\mathcal{L} = \langle L, \wedge, \vee \rangle$ is an orthocomplemented lattice, then the De Morgan's laws hold:*

L_{19} $\sim (a \wedge b) = (\sim a \vee \sim b)$;

L_{20} $\sim (a \vee b) = (\sim a \wedge \sim b)$.

Besides, in this case, $\sup\{a, b\}$ is defined if, and only if, $\inf\{a, b\}$ is also defined.

Definition 1.18. Let $\mathcal{L} = \langle L, \wedge, \vee \rangle$ be an orthocomplemented lattice. The element a is *orthogonal* to b, what is denoted by $a \perp b$, when:

$$a \perp b \Leftrightarrow a \leq \sim b.$$

Since $\mathcal{L}$ is an orthocomplemented lattice, so:

$$a \perp b \Leftrightarrow a \leq \sim b \Rightarrow \sim\sim b \leq \sim a \Leftrightarrow b \leq \sim \Leftrightarrow b \perp a.$$

Thus, the relation of orthogonality $\perp$ is symmetric.

Lemma 1.19. *In any lattice $\mathcal{L} = \langle L, \wedge, \vee \rangle$ we have that:*

L_{21} $x \leq z \Rightarrow x \vee (y \wedge z) \leq (x \vee y) \wedge z$.

Definition 1.20. If $x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$, then (y, z) is a *modular pair*. A lattice $\mathcal{L}$ is *modular* if every two elements of $\mathcal{L}$ determine a modular pair, that is, if y and z are in $\mathcal{L}$ then (y, z) and (z, y) are modular pairs.

Thus, every distributive lattice is a modular lattice.

Definition 1.21. Let $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ be an orthocomplemented lattice. Given $a, b, \in L$, the *orthomodular property* of a and b is defined by:

$$a \leq b \Rightarrow b = a \vee (b \wedge \sim a).$$

Theorem 1.22. *In any modular lattice $\mathfrak{L} = \langle L, \wedge, \vee\rangle$, the orthomodular property holds.*

Proof: *Let $a \leq b$. Since $\mathfrak{L}$ is a modular lattice, then $(\sim a, b)$ is a modular pair. Hence, $a \vee (b \wedge \sim a) = a \vee (\sim a \wedge b) = (a \vee \sim a) \wedge b = 1 \wedge b = b$.* ■

Definition 1.23. Let $\mathfrak{L} = \langle L, \wedge, \vee\rangle$ be an orthocomplemented lattice. The lattice $\mathfrak{L}$ is *orthocomplete* if for any pair a, b of orthogonal elements of L the supremum of $\{a, b\}$ is in L. If every pairwise orthogonal countable subset of L has a supremum, the lattice $\mathfrak{L}$ is *σ-orthocomplete*.

Definition 1.24. $\mathfrak{L}$ is an *orthomodular* poset if it is orthocomplete and for all $a, b \in L$ the orthomodular property holds.

The modular property is a particular case of distributivity because in a Boolean algebra $\mathfrak{B} = \langle B, 0, 1, \sim, \wedge, \vee\rangle$, if $a, b \in B$ and $a \leq b$, then:

$$b = b \wedge 1 = b \wedge (a \vee \sim a) = (b \wedge a) \vee (b \wedge \sim a) = a \vee (b \wedge \sim a),$$

that is, the modular property holds.

Definition 1.25. A *homomorphism* from a lattice $\mathfrak{L} = \langle L_1, \wedge, \vee\rangle$ into a lattice $\mathfrak{L}' = \langle L_2, \wedge, \vee\rangle$ is a function h from L_1 into L_2 such that:

$$h(x \wedge y) = h(x) \wedge h(y) \quad \textit{and} \quad h(x \vee y) = h(x) \vee (y).$$

Every homomorphism of lattices preserves ordering, that is,

$$x \leq y \Leftrightarrow x \vee y = y \Rightarrow h(x \vee y) = h(y) \Leftrightarrow h(x) \vee h(y) = h(y) \Leftrightarrow h(x) \leq h(y).$$

Definition 1.26. An *isomorphism* of lattices is a bijective homomorphism of lattices.

2 Quasi-sets

In this section, we outline some of the most basic notions of quasi-set theory which will play an important role in the discussion that follows. Of course we should provide the paper with a cluster of definitions in quasi-set theory, but the space suggests us to remind the reader the Chap.7 of French and Krause 2006 and French and Krause 2010 for further details. We will not present all the postulates and definitions of the theory, so we just revise some of the main ideas and results which are important for this paper.

Intuitively speaking, a quasi-set or qset is a collection of objects such that some of them may be indistinguishable without turning out to be identical. Of course this is not a strict 'definition' of a quasi-set, but act more or less as Cantor's 'definition' of a set as "any collection into a whole M of definite and separate, that is, distinguishable objects m of our intuition or our thought" serving just to provide an intuitive account of the concept. For detail we recommend the discussion in (French & Krause 2006). By 'indistinguishable' we mean agreement with respect to all properties, and in saying that a and b are 'identical' we mean intuitively speaking that they are *the very same entity*. The definitions of these concepts depend on the employed language and logic, but here we consider only their informal meanings.

The quasi-set theory $\mathfrak{Q}$ has in its main motivations some considerations taken from quantum physics, mainly in considering Schrödinger's idea that the concept of identity does not make sense when applied to elementary particles in orthodox quantum mechanics (Schrödinger 1952, p. 17-18). Another motivation, in our opinion, is the need, stemming from philosophical difficulties of dealing with collections of absolutely indistinguishable items that would be not 'the same' entity.[1] Of course, from a formal point of view, $\mathfrak{Q}$ can also be formally developed independently of any intended interpretation, but here we shall always keep in mind this 'quantum' motivation since, after all, it is the intended interpretation that has motivated the development of the theory.

The first point is to guarantee that identity and indistinguishability (or indiscernibility) will not collapse into one another when the theory is formally developed. We of course could just take an equivalence relation (or a congruence) within a standard set theory such as ZF to mimic indiscernibility, but this is just what we don't want to do; instead, we wish to deal with 'legitimate' (in a sense of the word to be explicated below) indistinguishable objects. Thus, we assume that identity, that will be symbolized by '$=$', is not a primitive relation, but that the theory has a weaker concept of indistinguishability, symbolized by '$\equiv$'. This is just an equivalence relation and holds among all objects of the considered domain. The ur-elements of the domain are divided up into two classes of objects, the m-objects, that stand for 'micro-objects', and M-objects, for 'macro-objects'.[2] Quasi-sets are those objects of the domain which are not ur-elements. Identity is defined for M-objects and 'sets' (entities that obey the primitive predicate Z) only. Thus, if we take just the part of theory obtained by ruling out the m-objects and collections (quasi-sets) whose have m-objects in their transitive

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[1] This is of course a way of speech. Despite some interpretations (such as Bohm's) presuppose an ontology similar to that of classical physics, in the sense of dealing with individuals, we shall keep here with the mainstream account of assuming that quantum objects may be 'absolutely indiscernible' in certain situations.

[2] Within $\mathfrak{Q}$, there are no explicit relations among m and M atoms, but we guess that the theory could be supplemented by mereological axioms enabling us to say that the M objects can be 'formed' by m objects in some sense. But this is still a work to be done.

closure, we obtain a copy of ZFU (ZF with ur-elements); if we further eliminate the M-objects, we get just a copy of the 'pure' ZF.

Indiscernible m-objects are termed non-individuals by historical reasons (French & Krause 2006). From the axioms of the theory $\mathfrak{Q}$ we can form collections of m-objects which may have a cardinal, termed its *quasi-cardinal*, but not an associated ordinal. Thus, the concepts of ordinal and cardinal are independent, as in some formulations of ZF proper, so, there are quasi-sets that cannot be ordered. Informally speaking, there may be quasi-sets of m-objects such that its elements cannot be identified by names, counted, ordered, although there is a sense in saying that these collections have a cardinal which cannot be defined in terms of ordinals. It is just by using quasi-cardinals that we can say (within $\mathfrak{Q}$) that a quasi-set has 'more than one' element. This discourse is of course dubious if we realize that the notion of identity does not hold in certain situations, but here the number (the quasi-cardinal) is what counts, and this resembles the Fock space formalism –see Teller (1995).

It is important to remark that, when $\mathfrak{Q}$ is used in connection with quantum physics, the m-objects are thought of as representing quantum entities (henceforth q-objects), but they are not necessarily 'particles' in the standard sense. Generally speaking, whatever 'objects' sharing the property of being indistinguishable can also be values of the variables of $\mathfrak{Q}$. For a survey of the various different meanings that the word 'particle' has acquired in connection with quantum physics see (Falkenburg 2007, chap.6).

Another important feature of $\mathfrak{Q}$ is that standard mathematics can be developed using its resources, because the theory is conceived in such a way that ZFU (and hence also ZF, perhaps with the axiom of choice, ZFC) is a subtheory of $\mathfrak{Q}$. In other words, the theory is constructed so that it extends standard Zermelo-Fraenkel with ur-relements (ZFU); thus standard sets of ZFU must be viewed as particular qsets, that is, there are qsets that have all the properties of the sets of ZFU, and the objects of $\mathfrak{Q}$ that correspond to the ur-elements of ZFU are identified with the M-atoms of $\mathfrak{Q}$. To make the distinction, the language of $\mathfrak{Q}$ encompasses a unary predicate Z such that $Z(x)$ says that x is (a copy of) a set of ZFU.

It is also possible to show that there is a translation from the language of ZFU into the language of $\mathfrak{Q}$, so that the translations of the postulates of ZFU become theorems of $\mathfrak{Q}$; thus, there is a 'copy' of ZFU in $\mathfrak{Q}$, and we refer to it as the 'classical' part of $\mathfrak{Q}$. In this copy, all the usual mathematical concepts can be stated, as for instance, the concept of ordinal (for Z-sets). This 'classical part' of $\mathfrak{Q}$ plays an important role in the formal developments of the next sections.

Furthermore, it should be recalled that the theory is constructed so that the relation of indiscernibility, when applied to M-atoms or Z-sets, collapses into standard identity of ZFU. The Z-sets are qsets whose transitive closure, as usually defined, does not contain m-atoms or, in other words, they are constructed in the classical part of the theory as in the figure (1).

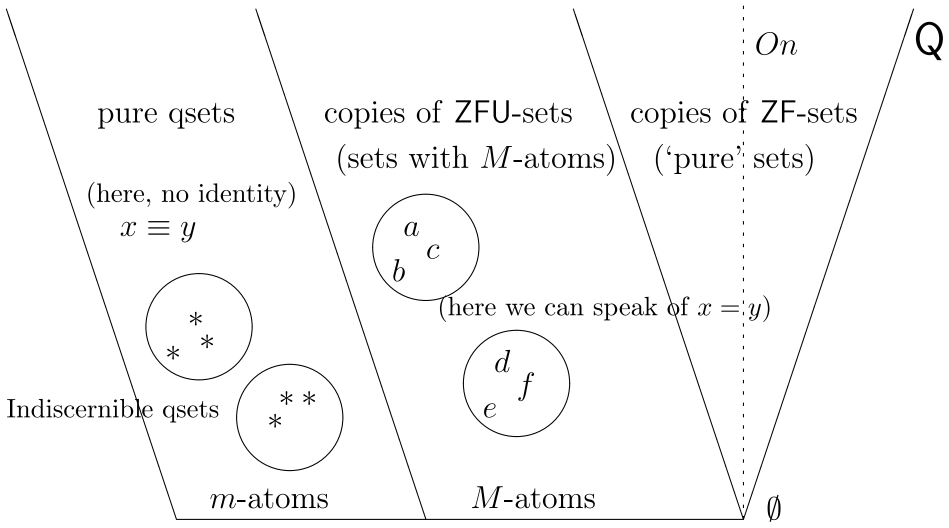

Figure 1: The Quasi-Set Universe Q: On is the class of ordinals, defined in the classical part of the theory.

In order to distinguish between Z-sets and qsets that may have m-atoms in their transitive closure, we write (in the metalanguage) $\{x : \varphi(x)\}$ for the former and $[x : \varphi(x)]$ for the latter. In $\mathfrak{Q}$, we term 'pure' those qsets that have only m-objects as elements (although these elements may be not always indistinguishable from one another, that is, the theory is consistent with the assumption of the existence of different kinds of m-atoms, i.e., not all of them must be indiscernible from one each other).

The concept of *extensional identity*, as said above, is a defined notion, and it has the properties of standard identity of ZFU. More precisely, we write $x = y$ (read 'x and y are extensionally identical') iff they are both qsets having the same elements (that is, $\forall z(z \in x \Leftrightarrow z \in y)$) or they are both M-atoms and belong to the same qsets (that is, $\forall z(x \in z \Leftrightarrow y \in z)$).

Since m-atoms my be indiscernible, in general is not possible to attribute an ordinal to collections of m-atoms. As a consequence, for these collections it is not possible to define the notion of cardinal number in the usual way, that is, through ordinals.[3] In the version of the theory, to remedy this situation, we admit also a primitive concept of quasi-cardinal which intuitively stands for the 'quantity' of objects in a collection.[4] The axioms for this notion grant that certain quasi-sets x, in particular, those whose elements are m-objects, may have a quasi-cardinal, written $qc(x)$, even when it is not possible to attribute an ordinal to them.

To link the relation of indistinguishability with qsets, the theory also encompasses an 'axiom of weak extensionality', which states, informally speaking, that those quasi-sets that have the same quantity, expressed by the quasi-cardinals, of elements of the same sort (in the sense that they belong to the same equivalence class of indistinguishable objects) are indistinguishable by their own. One of the interesting consequences of this axiom is related to the quasi-set version of the non observability of permutations in quantum physics, which is one of the most basic facts regarding indistinguishable quanta (French & Rickles 2003). In brief, remember that in standard set theories, if $w \in x$, then

$$(x - \{w\}\ ) \cup \{z\} = x \Leftrightarrow z = w.$$

We can 'exchange' (without modifying the original arrangement) two elements iff they are *the same* elements, by force of the axiom of extensionality. In contrast, in $\mathfrak{Q}$ we can prove the following theorem, where $[[z]]$, and similarly $[[w]]$, stand for a quasi-set with quasi-cardinal 1 whose only element is indistinguishable from z and, respectively, from w (the reader should not think that this element *is identical to either* z or w, because the relation of equality does not apply to these items; the set theoretical operations can be understood according to their usual definitions):

Theorem 2.1 (Unobservability of Permutations)**.** *Let x be a finite quasi-set such that x does not contain all indistinguishable from z, where z is an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists $[[w]]$ such that:*

$$(x - [[z]]\ ) \cup [[w]] \equiv x.$$

The theorem works to the effect that, supposing that x has n elements, then if we 'exchange' their elements z by corresponding indistinguishable elements w (set theoretically, this means performing the operation $(x - [[z]]\ ) \cup [[w]]$), then the resulting quasi-set remains *indistinguishable* from the one we started with. In a certain sense, it does not matter whether we are dealing with x or with $(x - [[z]]\ ) \cup [[w]]$. So, within $\mathfrak{Q}$, we can express that 'permutations are not observable',

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[3] We just recall that an ordinal is a transitive set which is well-ordered by the membership relation, and that a cardinal is an ordinal α such that for no $\beta < \alpha$ there does not exist a bijection from β to α (Devlin 1993).

[4] The notion of quasi-cardinal can be defined for finite quasi-sets in Domenech & Holik 2007, and independently by Arenhart 2008.

without necessarily introducing symmetry postulates, and in particular we derive 'in a natural way' the quantum statistics (Krause *et alli* 1999; French, Krause 2006, Ch. 7). Further applications to the foundations of quantum mechanics can be seen in (Domenech *et alli* 2008, 2010).

3 Clouds

In this section we shall be working within the theory $\mathfrak{Q}$, and we will emphasize the case that involves m-atoms only. Here we follow Krause (2005), with slight modifications. Thus, keeping with quasi-set theory in mind, we begin with a characterization of certain relational structures in which the involved relations do not depend on particular elements. It should be recalled that in standard extensional set theories (say, in ZF with regularity), a relation is a set of n-tuples of elements. Thus, the relation is 'formed' only after the 'formation' of its elements. In quasi-set theory, due to the indiscernibility of some elements, we can form 'quasi-relations' which remain 'unaltered' when its elements are exchanged by indiscernible ones. The analogy with chemistry is immediate; a chemical compound, say a water molecule H_2O 'remains unaltered' if by some device the particular atoms are changed by other of similar species (we mean, H atoms changed by H atoms and O atoms changed by O atoms). The same happens in a process of ionization; let us suppose an Hydrogen atom in its fundamental state. After an ionization process, an electron may be realized, getting a negative ion H^-. But we can capture an electron getting a neutral atom again. For all the physical purposes, the two neutral atoms, the original one and that obtained after the process, are absolutely indiscernible, and the same happens with the electrons which were realized and captured. We can express this fact in $\mathfrak{Q}$ as follows.

A quasi-relation on a qset A is a qset R whose elements are ordered 'pairs' that belong to A. These 'pairs' must be understood in the right way (in terms of the postulates of $\mathfrak{Q}$). Since the identity relation is not defined for m-atoms, it cannot be used here, so an ordered 'pair' $\langle z, w, \rangle$ becomes something like the collection (qset) of the indistinguishable from z that belong to A, denoted by $[z]$, and the collection of the indistinguishable from either z or w, denoted by $[z, w]$, that belong to A; in symbols, $\langle z, w, \rangle := [[z], [z, w]]$, which resembles Wiener-Kuratowski's definition ordered pair. So, each 'pair' may contains more than two elements (the word 'pair' here looks like 'pair of kinds'). So, a (binary) quasi-relation R on A is a qset which obeys the following predicate $\Re$:

$$\Re(R) := \forall z(z \in R \to \exists u \exists v(u \in A \land v \in A \land z = \langle u, v \rangle)).$$

Then, in an analogy with the chemical examples mentioned above, we can formulate the following

QUESTION: given a certain n-ary q-relation R on a pure qset A, if $R(x_1, \ldots, x_n)$ holds, does $R(x'_1, \ldots, x'_n)$ also hold when $x_i \equiv x'_i$? In other words, are the relations 'preserved' when the relata are exchanged by indistinguishable ones?

The first and direct answer to the above question is that it depends on the relation. If R is membership, then the intended result fails, because if $x \in y$ and $x \equiv x'$, $y \equiv y'$, then nothing in the axioms of the theory $\mathfrak{Q}$ entails that $x' \in y'$. This is one of the basic results that make the primitive relation of indistinguishability distinct from identity.[5] Membership is the only primitive relation of $\mathfrak{Q}$ which does not enable substitutivity by indistinguishable (French, Krause 2006, Ch. 7). So, let us take R to be whatever relation distinct from membership; furthermore, we shall work with binary relations only for simplicity and we will be paying attention to relations on qsets whose elements are m-atoms only. So, we can reformulate the above question: if R is a binary relation, distinct from membership, and if $R(x, y) \wedge x' \equiv x \wedge y' \equiv y$, does this entail that $R(x', y')$ holds as well? The most interesting case is of course when both x and y are m-atoms, because if there are no m-atoms involved, then $\equiv$ becomes the extensional identity and the answer is a straightforward yes. Thus, there is in whatever structure built in $\mathfrak{Q}$ a 'natural' automorphism, namely, the q-function which associates to a certain element of the domain one indistinguishable one. If the element is 'classical' (an M-atom or a set), then of course this automorphism is the identity function, but it is not in the case of m-atoms.

Let us fix a finite domain (a pure qset) D and suppose for simplicity that R is defined on $A \subseteq D$, which is enough for our purposes. $R(x, y)$ means $\langle x, y \rangle \in R$, that is, $[[x], [x, y]] \in R$, where $[x]$ is the qset of all indistinguishable from x that belong to A, which may contain more than one element), and $[x, y]$ is the qset of the indistinguishable of either x or y that belong to A. Furthermore, x and y are not playing the role of names for objects of the domain; they act as generalized names instead, meaning something like 'some' indistinguishable from x or y respectively. So, a binary relation in the theory $\mathfrak{Q}$ is not a well 'defined' (by its extension) collection of ordered pairs of the elements of some set. If $R(x, y)$ holds, we are not saying that *that* specific x and *that* specific y are in the relation (since we cannot express these entities–as individuals–in the language), but that *some* indistinguishable from x is in the relation with *some* indistinguishable from y are in the relation. The problem now is to explain in what sense, when R is defined on a certain A, if x' and y' are indistinguishable respectively from x and y, we can ensure that $R(x, y)$ is true, the same happens with $R(x', y')$ (for x' and

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[5] In ZF, if $x \in y$ and $x = x'$, then $x' \in y$; if $y \in x$ and $x = x'$, then $y \in x'$.

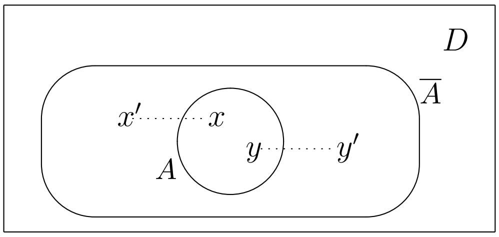

Figure 2: The quasi-set A and its cloud related to D.

y' may be not members of A). So, the apparent answer to our Question above would be in the negative.

But there is a sense according to which we can circumvent this 'no', that is, the answer to the above claim can be an 'yes'. It is enough to consider what we shall term the *surroundings* of the qset A, or the *cloud* of A, denoted $\overline{A}$. The cloud of A is defined relatively to the qset D that contains A as follows:

$$\overline{A} := [y \in D : y \equiv x \wedge x \in A].^{6}$$

In words, the cloud of A relative to D is the qset of the elements of D which are indistinguishable from the elements of A. Intuitively, $\overline{A}$ acts as the surroundings from where A can 'exchange' elements (Figure 2). Let us suppose that $\widehat{R}$ is the extension of R to $\overline{A}$, that is $\widehat{R}$ is the qset of all 'pairs' $\langle x, y\rangle$ with x and y in the $\overline{A}$ such that $\langle x, y\rangle \in R$ when $x, y \in A$. Then we can prove in $\mathfrak{Q}$ the following result:

Theorem 3.1. *If $A \subseteq D$, $x, y \in A$ and $R(x, y)$, where R is a quasi-relation on A, then there exist $x', y' \in D - A$ such that $x' \equiv x$, $y' \equiv y$ and $\widehat{R}(x', y')$.*

Proof: *If $\neg\widehat{R}(x', y')$, then $[[x'], [x', y']] \notin \widehat{R}$, hence $[x'] \notin \widehat{R}$ or $[y'] \notin \widehat{R}$ or both. But if $[x'] \notin \widehat{R}$, then $[x] \notin R$, against the hypothesis. The same for $[y']$.* ■

Intuitively, the theorem says that if $R(x, y)$ holds for $x, y \in A$, then if x' and y' are indistinguishable from x and y respectively and belong to a qset D which includes A, then $\widehat{R}$ holds for these elements, that is, $\widehat{R}(x', y')$ holds. We remark that there would be no mathematical sense in saying, in the general case, that $R(x', y')$ holds, for x' and y' may do not belong to A, and R is a quasi-relation defined on A. The extension $\widehat{R}$ of R plays the role of R for the elements of the cloud of A and coincides with R within A. So, in saying that $\widehat{R}(x', y')$ holds, we are in a certain sense granting that the relation R is maintained (though $\widehat{R}$) when the elements it relates are exchanged by suitable indistinguishable ones (say, taken from its neighborhood, like a measurement apparatus in a physical experience), and hence it does not depend on the particular relata it relates, as

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[6] Taking some lessons from standard algebra, we could call the cloud an i-orbit, the i coming to express indiscernibility. But we shall keep with our original terminology.

they were individuals. It seems to us that this is precisely what the chemical situations involving ionization and others among the above exemplified cases are suggesting us.

Next, we introduce, still in $\mathfrak{Q}$, the following definition. Intuitively speaking, we can regard x as a *potential element* of y when there is in y one element indistinguishable from x; let us write $x \lhd y$ to represent that, that is,

Definition 3.2. $x \lhd y := \exists z(z \in y \land z \equiv x)$.

Thus, its negation $x \ntriangleleft y$ reads $\forall z(x \in y \to z \not\equiv x)$, that is, x is not a potential element of y when there are no indiscernible from x that belong to y. By using this concept, we can introduce *the fuzzy complement* of x relative to a certain previously given qset z, with $x \subseteq z$, as the collection (qset) of the elements of z for which we can not prove that they are not potential elements of x. This is precisely the cloud of x relative to z, namely, which can be written alternatively as being the qset (see definition 4.7) $\overline{x}_z := [t \in z : t \lhd x]$.

The concept of cloud of a qset suggests the idea that a qset x is something *in between* its extension $Ext(x)$, namely, the qset of the objects that belong to x and its cloud $\overline{x}_z$ for some z, the qset of its 'potential' elements. Due to this fact that the elements of a pure qset do not have well defined identity criteria, we may say that a qset is not strictly determined by its elements, hence, some degree of intentionality is also present here. Thus, quasi-set theory enables us to interpret the collections from both perspectives: an intensional one and an extensional one.

Since the m-atoms do not have identity criteria, then there is still a certain epistemic indeterminacy whether a certain element does belong to a certain qset or not; all we can say is that there may be traces of something which behaves like such an element in the collection (when there is some indistinguishable from it in the qset), but we can never prove that a certain element is exactly *that* element, although we can formulate an informal discourse about an element whatever, as physicists usually do. Really, we have elements (this is particularly important for non-individual entities) which 'could be' in x, but we can not prove that they really are or are not elements of x.

Of course, the idea of the cloud of x may suggest several applications. In the next section we shall see some of them.

4 The quasi-lattice $\mathfrak{I}$

We recall that the typical algebraic structure arising from the mathematical formalism of quantum mechanics is not a Boolean algebra, but an orthocomplete

(σ-orthocomplete in the general case) and orthomodular lattice. We shall see from now on that in quasi-set theory, by considering indiscernibility right from the start, a similar structure ‘naturally’ arises.

Let us begin by considering the concepts of Tarski’s system and that of topological space.

Definition 4.1. A *Tarski’s space* (*deductive system* or *closure space*) is a pair $\mathcal{T} = \langle E, ^- \rangle$ where E is a nonempty set and $^-$ is a function $^- : \mathcal{P}(E) \to \mathcal{P}(E)$, called the *Tarski’s consequence operator*, such that:

(i) $A \subseteq \overline{A}$;
(ii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$;
(iii) $\overline{\overline{A}} \subseteq \overline{A}$.

Theorem 4.2. *If $\langle E, ^- \rangle$ is a Tarski’s space, then the following holds:*

(i) $\overline{\overline{A}} = \overline{A}$*;*
(ii) $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$*;*
(iii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$*;*
(iv) $\overline{\overline{A} \cup \overline{B}} = \overline{A \cup B}$*;*
(v) $\overline{A} \cap \overline{B} = \overline{\overline{A} \cap \overline{B}}$.

Proof: See (Feitosa, Grácio, Nascimento 2007). ■

Definition 4.3. Let $\langle E, ^- \rangle$ be a Tarski’s Space and $A \subseteq E$. The set A is *closed* in $\langle E, ^- \rangle$ when $\overline{A} = A$ and A is *open* in $\langle E, ^- \rangle$ when its complement relative to E, denoted by A^C, is closed.

Definition 4.4. Let $\langle E, ^- \rangle$ be a Tarski’s Space and $A \subseteq E$. The set $\overline{A}$ is the *closure* of A and the set $\mathring{A} = (\overline{A^C})^C$ is the *interior* of A. The frontier of A, denoted by ∂A, is the set $\partial A = \overline{A} - \mathring{A}$.

Theorem 4.5. *In any Tarski’s Space $\langle E, ^- \rangle$ it follows that: $\mathring{A} \subseteq A \subseteq \overline{A}$.*

Proof: Immediate from the above definition. ■

Definition 4.6. A *topological space* is a pair $\langle E, ^- \rangle$ where E is a nonempty set and $^-$ is a function $^- : \mathcal{P}(E) \to \mathcal{P}(E)$, such that:

(i) $A \subseteq \overline{A}$;
(ii) $\overline{\overline{A}} \subseteq \overline{A}$;
(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
(iv) $\overline{\emptyset} = \emptyset$.

In any topological space it is immediate to observe that the condition $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$ holds, since $A \subseteq B \Rightarrow A \cup B = B \Rightarrow \overline{A} \subseteq \overline{A} \cup \overline{B} = \overline{A \cup B} = \overline{B}$.

From now on we shall be working within $\mathfrak{Q}$. Now we use the concept of cloud, as pointed out in the previous section, in order to delineate the algebraic aspects of indiscernible elements. By simplicity of notation, we shall term $\overline{A}$ (instead of

$Cloud_U(A)$) the cloud of A relative to a universal qset U. Recalling what was said at page 13 about the cloud of a quasi-set, we introduce the following definition:

Definition 4.7. Let U be a non empty qset and A be a subqset of U. The *cloud* of A in U is the qset:

$$\overline{A} = [y \in U : x \equiv y \wedge x \in A]$$

Intuitively speaking, $\overline{A}$ is the qset of the elements of U, the universe, which are indistinguishable from the elements of A. If A is a *set* in $\mathfrak{Q}$ (we shall say that these qsets are 'Z-sets'), that is, a copy of a set of ZFU, then of course the only indistinguishable of a certain x is x itself and $\overline{A} = A$.

In $\mathfrak{Q}$ the concept of function must be generalized, because if there are m-atoms involved, a mapping in general does not distinguish between arguments and values. Thus we use the notion of q-function, which leads indistinguishable objects into indistinguishable objects, and which reduces to standard functions when there are no m-atoms involved. Thus, from the formal point of view, the defined mapping may associate to A whatever qset from a collection of indistinguishable qsets.

Theorem 4.8. *The application that associates to every subqset of U its cloud is a Tarski's operator and $\langle U, ^- \rangle$ is a Tarski's Space.*

Proof: (i) $A \subseteq \overline{A}$: If $t \in A$, by the reflexivity of $\equiv$, we have that $t \equiv t$, hence $t \in \overline{A}$;

(ii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$: Let $A \subseteq B$. If $t \in \overline{A}$, then there exists $x \in A$ such that $t \equiv x$. Since $x \in B$, then $t \in \overline{B}$;

(iii) $\overline{\overline{A}} \subseteq \overline{A}$: If $t \in \overline{\overline{A}}$, then there exists $x \in \overline{A}$ such that $t \equiv x$. But then there exists $y \in A$ such that $x \equiv y$ and, by the transitivity of $\equiv$, it follows that $t \equiv y$, and hence $t \in \overline{A}$. ■

From now on, we shall suppose that U is a nonempty closed qset and for $A \subseteq U$, $\overline{A}$ is the cloud of A relative to U. Hence, by hypothesis, U contains all the indistinguishable objects of its own elements (this can be seen as a working hypothesis that avoids us to refer to another qset containing U and so on). Some interpretations linked to physical situations are possible. For instance, $\overline{A}$ can be thought as the region where the wave function A of a certain physical system is different from zero. Another possible interpretation is to suppose that the clouds describe the systems plus the cloud of virtual particles that accompany those of the considered system. But in this paper we shall be not considering these motivations, but just to explore its algebraic aspects.

It is immediate to prove the following theorem:

Theorem 4.9. *$\langle U, ^- \rangle$ is a topological space.*

Proof: Since $\langle U, ^- \rangle$ is a Tarski's Space, we only need to prove:

(i) $\overline{A} \cup \overline{B} = \overline{A \cup B}$: As $A \subseteq A \cup B$, so $\overline{A} \subseteq \overline{A \cup B}$. In the same way, $B \subseteq A \cup B$

and $\overline{B} \subseteq \overline{A \cup B}$. Thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Conversely, suppose that $t \in \overline{A \cup B}$. Then there is $x \in A \cup B$ such that $t \equiv x$, that is, $t \equiv x$ and $x \in A$ or $x \in B$. In this way $t \in \overline{A}$ or $t \in \overline{B}$ and therefore $t \in \overline{A} \cup \overline{B}$;
(ii) $\overline{\emptyset} = \emptyset$: it follows immediately from the definition of cloud. ■
Since $\langle U, ^{-} \rangle$ is a topological space, then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Example:
(a) Let $U = \{a, b, c\}$ such that the quasi-cardinal of U is 3 , with $b \equiv c$ but $a \not\equiv b$ and $A = \{a, b\}, B = \{b\}$ and $C = \{c\}$. In this case we have that $\overline{\emptyset} = \emptyset$, $\overline{A} = U$, $\overline{B} = \{b, c\} = \overline{C}$ and even as $A \cap C = \emptyset$, we have $\overline{A} \cap \overline{C} = \overline{C} \neq \emptyset$. Hence, we observe that $\overline{A \cap C} \neq \overline{A} \cap \overline{C}$.

The next definition introduces operations on subqsets of an universe U.

Definition 4.10. Let $A, B \subseteq U$. The $\mathfrak{I}$-*lattice operations* are defined by:
($\sqcap$): $A \sqcap B = A \cap B$
($\sqcup$): $A \sqcup B = \overline{A \cup B}$
($\mathbf{0}$): $0 = \emptyset$
($\mathbf{1}$): $1 = U$.

Like in the usual quantum algebra, the motivation for the definition of $\sqcap$ and $\sqcup$ comes from the fact that there the intersection of two closed subspaces of a Hilbert Spaces is a closed subspace and here the intersection of clouds is a cloud, but there in general the union is not a Hilbert Space, so it is considered the generated space by the two given spaces; and here we take the cloud generated by the union of the two qsets.

Now we can caracterize the lattice of indiscernible elements of $\mathcal{Q}$.

Theorem 4.11. *In the structure* $(\mathcal{P}(U), \sqcap, \sqcup, 0, 1)$ *the following holds:*
Associativity: $A \sqcap (B \sqcap C) = (A \sqcap B) \sqcap C$ *and* $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$*;*
Commutativity of $A \sqcap B = B \sqcap A$ *and* $A \sqcup B = B \sqcup A$*;*
Absorption: $A \sqcap (A \sqcup B) = A$ *e* $A \sqcup (A \sqcap B) = \overline{A}$*;*
Zero $0 = \emptyset$*:* $\emptyset \sqcap A = \emptyset$ *and* $\emptyset \sqcup A = \overline{A}$*;*
One $1 = U$*:* $U \sqcap A = A$ *and* $U \sqcup A = \overline{A}$.

Proof: Since the operation $\sqcap$ is defined by the usual intersection $A \sqcap B = A \cap B$, then it is associative and commutative. The commutativity of $\sqcup$ is immediate in the two cases.
Associativity of $\sqcup$: Given $A, B, C \in \mathcal{P}(U)$ it follows that: $A \sqcup (B \sqcup C) = A \sqcup \overline{(B \cup C)} = A \sqcup (\overline{B} \cup \overline{C}) = \overline{A \cup (\overline{B} \cup \overline{C})} = \overline{A} \cup \overline{(\overline{B} \cup \overline{C})} = \overline{A} \cup (\overline{B} \cup \overline{C}) = (\overline{A} \cup \overline{B}) \cup \overline{C} = \overline{(A \cup B)} \cup \overline{C} = \overline{\overline{(A \cup B)}} \cup \overline{C} = \overline{\overline{(A \cup B)} \cup C} = \overline{(A \cup B)} \sqcup C = (A \sqcup B) \sqcup C$.

Commutativity of $\sqcup$: Given $A, B \in \mathcal{P}(U)$ it follows that: $A \sqcup B = \overline{A \cup B} = \overline{B \cup A} = B \sqcup A$.
Absorption: $A \sqcap (A \sqcup B) = A \cap (A \sqcup B) = A$, because $A \subseteq A \sqcup B$;
$A \sqcup (A \sqcap B) = A \sqcup (A \cap B) = \overline{A \cup (A \cap B)} = \overline{A}$.
Zero: $0 \sqcap A = \emptyset \cap A = \emptyset = 0$. And $0 \sqcup A = \overline{\emptyset \cup A} = \overline{A}$.
One: $A \sqcap 1 = A \cap U = A$. And $A \sqcup 1 = \overline{A \cup U} = \overline{U} = U = 1$. ■

Since $A \sqcup (A \sqcap B) = \overline{A}$ and $0 \sqcup A = \overline{A}$, the structure $(\mathcal{P}(U), \sqcap, \sqcup, 0, 1)$ is not a lattice, but a particular variation of lattice.

Theorem 4.12. *For every $A, B \in \mathcal{P}(U)$ the following holds:*
(i) $\overline{A} \subseteq A \sqcup B$ and $\overline{B} \subseteq A \sqcup B$;
(ii) $A \sqcap B \subseteq A$ and $A \sqcap B \subseteq B$;
(iii) $A \sqcup B = \overline{A} \sqcup \overline{B} = \overline{A \sqcup B}$ and $A \sqcap B \subseteq \overline{A \sqcap B} \subseteq \overline{A} \sqcap \overline{B}$;
(iv) $\overline{A \sqcap B} \subseteq A \sqcup B$;
(v) $A \sqcap A = A$;
(vi) $A \sqcup A = \overline{A}$.
PROOF: Immediate. ■

Example:
(a) Let $U = \{a, b, c, d\}$ such that $a \equiv d$, $b \equiv c$, but $a \not\equiv b$, $A = \{a, b\}$ and $B = \{a, c\}$. Hence, $A \sqcap B = A \cap B = \{a\}$, $\overline{A \sqcap B} = \{a, d\}$, $\overline{A} \sqcap \overline{B} = \overline{A} \cap \overline{B} = \{a, b, c, d\} \cap \{a, b, c, d\} = \{a, b, c, d\}$.

If A and B are closed, then $A \cup B$ and $A \cap B$ are closed too, $A \sqcap B = A \cap B$ and $A \sqcup B = A \cup B$.

Theorem 4.13. *Let $\mathcal{C}$ be the qset of all closed subqsets of U. Then the structure $\mathfrak{C} = \langle \mathcal{C}, \sqcap, \sqcup, 0, 1 \rangle$ is a distributive lattice with 0 and 1.*
PROOF: In this case, for every $A \subseteq U$, it holds that $\overline{A} = A$ and then we can see the operation $^{-}$ simply as the identity function and, of course, all the result are valid. ■

This result is not surprising, because in that case we would be dealing exactly with set theoretical operations which, defined on the closed qsets of U, act as the usual set theoretical properties on standard sets. But when we consider all qsets of U and not only the closed ones, some lattice laws and the distributive laws do not hold.

We can show that the structure $(\mathcal{P}(U), \sqcap, \sqcup, 0, 1)$ is, in general, not distributive.

Example:
(a) Let $U = \{a, b, c\}$ such that $a \not\equiv b \equiv c$, $A = \{a\}$, $B = \{b\}$ and $C = \{c\}$. The

structure $(\mathcal{P}(U), \sqcap, \sqcup, 0, 1)$ is not distributive, because:

$$A \sqcup (B \sqcap C) = A \sqcup \emptyset = A = \{a\} \text{ and}$$
$$(A \sqcup B) \sqcap (A \sqcup C) = U \sqcap U = U \neq \{a\}.$$

Since the corresponding structure $\mathfrak{I} = (\mathcal{P}(U), \sqcap, \sqcup, 0, 1)$ emerges from aspects of indiscernible elements and it has similarities with a lattice with 0 and 1, we propose to call it *quasi-lattice of indiscernibility*, or just $\mathfrak{I}$-quasi-lattice.

Others distinctive characteristics of this ‘quasi-lattice’ are obtained when we introduce other similar operations to those of order and complement.

As in $\mathfrak{Q}$ we have the relation of inclusion we could define an order relation as $\subseteq$, but in that case we can not use the operations to caracterize that ordering. So we can introduce another order into $\mathfrak{I}$.

Definition 4.14. (The $\mathfrak{I}$-*order* of $\mathfrak{I}$) $A \sqsubseteq B \Leftrightarrow A \sqcup B = \overline{B}$.

Theorem 4.15. *The $\mathfrak{I}$-order obeys the following properties:*

(i) $A \sqsubseteq B \Leftrightarrow \overline{A} \subseteq \overline{B}$;

(ii) $A \sqsubseteq A$, $A \sqsubseteq \overline{A}$ *and* $\overline{A} \sqsubseteq A$;

(iii) $A \sqsubseteq B$ *and* $B \sqsubseteq A \Rightarrow \overline{A} = \overline{B}$;

(iv) $A \sqsubseteq B$ *and* $B \sqsubseteq C \Rightarrow A \sqsubseteq C$.

Proof: (i) $A \sqsubseteq B \Leftrightarrow \overline{B} = A \sqcup B = \overline{A \cup B} = \overline{A} \cup \overline{B} \Leftrightarrow \overline{A} \subseteq \overline{B}$; (ii), (iii), and (iv) follow from (i) and the definition. ■

From the item (iii) of the previous theorem we observe that $\sqsubseteq$ is not exactly a parcial order as usual, but just an almost parcial order.

The order defined by $A \sqcap B = A \Leftrightarrow A \subseteq B$ is the usual ordering and hence distinct from the order $\sqsubseteq$.

Corollary 4.16. *The $\mathfrak{I}$-order obeys the following properties:*

(i) $\overline{A \sqcap B} \sqsubseteq A$;

(ii) $C \sqsubseteq A$ *and* $C \sqsubseteq B \Rightarrow C \sqsubseteq A \sqcap B$;

(iii) $\overline{A} \sqsubseteq A \sqcup B$;

(iv) $A \sqsubseteq C$ *and* $B \sqsubseteq C \Rightarrow A \sqcup B \sqsubseteq C$.

Proof: Immediate. ■

Relative to the complement we have two choose. We can define the complement as the usual complement of sets $A' = A^C$ whose is orthocomplete or a new complement as in the following.

Definition 4.17. The $\mathfrak{I}$-*complement* of the qset A is defined by:

$$A^{\perp} = U - \overline{A} = \overline{A}\,'.$$

Thus, the $\mathfrak{I}$-complement of a qset A relative to the universe U is a subqset of U, termed $A^{\perp}$, which has no indistinguishable element from any element of A, that is, in $A^{\perp}$ there are no elements indiscernible from the elements of $\overline{A}$, in accordance to definition of cloud.

Theorem 4.18. *Let $A, B \in \mathcal{P}(U)$. Then:*

(i) $\emptyset^{\perp} = U$;
(ii) $U^{\perp} = \emptyset$;
(iii) $U - A^{\perp} = (A^{\perp})' = \overline{A}$;
(iv) $\overline{A}^{\perp} = A^{\perp}$;
(v) $A^{\perp^{\perp}} = \mathring{A}$;
(vi) $A \sqsubseteq B \Rightarrow B^{\perp} \sqsubseteq A^{\perp}$.

Proof: (i) $\emptyset^{\perp} = U - \overline{\emptyset} = U - \emptyset = U$;

(ii) $U^{\perp} = U - \overline{U} = U - U = \emptyset$;

(iii) $U - A^{\perp} = U - (U - \overline{A}) = \overline{A}\,'\,' = \overline{A}$;

(iv) $\overline{A}^{\perp} = U - \overline{\overline{A}} = U - \overline{A} = A^{\perp}$;

(v) $A^{\perp^{\perp}} = \overline{A}\,'^{\perp} = \overline{A}\,'^{-}{}' = \mathring{\overline{A}} = \mathring{A}$;

(vi) $A \sqsubseteq B \Leftrightarrow A \sqcup B = \overline{B}$, hence $\overline{A} \cup \overline{B} = \overline{B}$ and $\overline{A} \subseteq \overline{B}$. But this implies that $\overline{B}\,' \subseteq \overline{A}\,'$, that is, $B^{\perp} \subseteq A^{\perp}$. So $B^{\perp} \cup A^{\perp} = A^{\perp}$, then $\overline{B^{\perp} \cup A^{\perp}} = \overline{A^{\perp}}$, hence $B^{\perp} \sqcup A^{\perp} = \overline{A^{\perp}}$ or $B^{\perp} \sqsubseteq A^{\perp}$. ■

The property (v) show us that the operation ${}^{\perp}$ is not exactly a complement over $\langle \mathcal{P}(U), \sqcap, \sqcup, {}', {}^{\perp}, 0, 1\rangle$. However, as we can see now, the operation ${}^{\perp}$ generate the least complement for A.

Lemma 4.19. *In an $\mathfrak{I}$-lattice it holds $\langle \overline{A} \cap \overline{B} \rangle\,' \subseteq \langle \overline{A \cap B} \rangle\,'$.*

Proof: $A \cap B \subseteq A \Rightarrow \overline{A \cap B} \subseteq \overline{A}$. From that we have $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and $(\overline{A} \cap \overline{B})\,' \subseteq (\overline{A \cap B})\,'$. ■

Theorem 4.20. *If $A, B \in \mathcal{P}(U)$, then:*

(i) $A \sqcup A^{\perp} = 1$;
(ii) $A \sqcap A^{\perp} = 0$;
(iii) $A \sqcup (B \sqcap B^{\perp}) = \overline{A}$;
(iv) $A \sqcap (B \sqcup B^{\perp}) = A$;
(v) $(A \sqcup B)^{\perp} = A^{\perp} \sqcap B^{\perp}$;
(vi) $A^{\perp} \sqcup B^{\perp} \sqsubseteq (A \sqcap B)^{\perp}$.

Proof: *(i) If $x \in U$, then $x \in \overline{A}$ or $x \in \overline{A}\,'$. If $x \in \overline{A}$, then $x \in \overline{A} \cup A^{\perp} \subseteq \overline{A} \cup \overline{A^{\perp}} = \overline{A \cup A^{\perp}} = A \sqcup A^{\perp}$. If $x \in \overline{A}\,' = A^{\perp}$, then $x \in \overline{A} \cup A^{\perp} \subseteq \overline{A} \cup \overline{A^{\perp}} = \overline{A \cup A^{\perp}} = A \sqcup A^{\perp}$. Hence, $U \subseteq A \sqcup A^{\perp}$ and since $A \sqcup A^{\perp} \subseteq U$, we have $U = A \sqcup A^{\perp}$.*

(ii) $A \sqcap A^{\perp} = A \sqcap \overline{A}\,' = A \cap \overline{A}\,' = \emptyset = 0$;

(iii) $A \sqcup (B \sqcap B^{\perp}) = A \sqcup 0 = \overline{A}$;

(iv) $A \sqcap (B \sqcup B^{\perp}) = A \sqcap 1 = A$;

(v) $(A \sqcup B)^{\perp} = \overline{A \cup B}^{\perp} = (\overline{A} \cup \overline{B})^{\perp} = \overline{(\overline{A} \cup \overline{B})}\ ' = \overline{(A \cup B)}\ ' = (\overline{A} \cup \overline{B})\ ' = \overline{A}\ ' \cap \overline{B}\ ' = A^{\perp} \sqcap B^{\perp}$;

(vi) $A^{\perp} \sqcup B^{\perp} = \overline{A^{\perp} \cup B^{\perp}} = \overline{\overline{A}\ ' \cup \overline{B}\ '} = \overline{(\overline{A} \cap \overline{B})\ '} \subseteq \overline{\overline{A \cap B}\ '} = \overline{(A \cap B)^{\perp}} = \overline{(A \sqcap B)}^{\perp}$. ■

The operation $\sqsubseteq$ of the $\mathfrak{I}$-lattice $\langle \mathcal{P}(U), \sqcap, \sqcup, ', ^{\perp}, 0, 1\rangle$ has the following $\mathfrak{I}$-orthomodular property:

$$A \sqsubseteq B \Rightarrow \overline{B} = A \sqcup (B \sqcap A^{\perp}).$$

Theorem 4.21. *Given* $A, B \in \mathcal{P}(U)$*:* $A \sqsubseteq B \Rightarrow \overline{B} = \overline{A} \sqcup (B \sqcap A^{\perp})$.

Proof: $\overline{A} \sqcup (B \sqcap A^{\perp}) = \overline{\overline{A} \cup (B \cap A^{\perp})} = \overline{(\overline{A} \cup B) \cap (\overline{A} \cup A^{\perp})} = \overline{(\overline{A} \cup B) \cap U} = \overline{(\overline{A} \cup B)} = \overline{\overline{A}} \cup \overline{B} = \overline{A} \cup \overline{B} = A \sqcup B = \overline{B}$. ■

Let $A, B \subseteq U$. The qset A is *orthogonal* to B, what is written $A \perp B$, if $A \sqsubseteq B^{\perp}$. Furthermore, a collection S of elements of $\mathcal{P}(U)$ is called *pairwise orthogonal* if for any $A, B \in S$ such that $A \neq B$, it results that $A \perp B$.

Theorem 4.22. *If* $A, B \in \mathcal{P}(U)$*, then:*

(i) $\overline{A} \subseteq \overline{B^{\perp}} \Leftrightarrow A \subseteq B^{\perp}$;

(ii) $A \perp B \Leftrightarrow A \sqcap \overline{B} = \emptyset$.

Proof: *(i)* $(\Rightarrow)$ *Suppose that there exists* $x \in A$ *such that* $x \notin B^{\perp}$*. Then* $x \in A \subseteq \overline{A} \subseteq \overline{B^{\perp}}$*. But* $x \equiv b \in B^{\perp} = \overline{B}\ '$ *implies that* $b \notin \overline{B}$ *and* $x \notin B^{\perp} = \overline{B}\ ' \Rightarrow x \in \overline{B} \Rightarrow b \in \overline{B}$*, that is a contradiction. Hence,* $A \subseteq B^{\perp}$.

$(\Leftarrow)$ *Immediate.*

(ii) $A \sqcap \overline{B} = \emptyset \Leftrightarrow A \cap \overline{B} = \emptyset \Leftrightarrow A \subseteq \overline{B}\ ' \Leftrightarrow A \subseteq B^{\perp} \Leftrightarrow \overline{A} \subseteq \overline{B^{\perp}} \Leftrightarrow \overline{A} \cup \overline{B^{\perp}} = \overline{B^{\perp}} \Leftrightarrow \overline{A \cup B^{\perp}} = \overline{B^{\perp}} \Leftrightarrow A \sqcup B^{\perp} = \overline{B^{\perp}} \Leftrightarrow A \sqsubseteq B^{\perp} \Leftrightarrow A \perp B$. ■

Intuitively speaking, $A \sqcap \overline{B} = \emptyset$ says that A has no indistinguishable element from the elements of B.

In quantum logic, the operations $\sqsubseteq$ and $^{\perp}$ are usually understood as an *implication* and a *negation* respectively. Thus, we may introduce the concept of *logical incompatibility* just using the idea of orthogonality (Dalla Chiara, Giuntini, Greechie 2004, p. 12): A is incompatible with B iff A implies the negation of B iff they are orthogonal. The negation of the relation $\perp$ is called *accessibility*, written $A \not\perp B$.

All of this show that our structure $\mathfrak{I}$ resembles an structure non-distributive with similarities of an orthocomplete orthonormal lattice, and it is a Boolean lattice if we consider only the closed qsets.

In the next section we begin an algebraic theory about the $\mathfrak{I}$-lattice.

5 Algebraic elements from the model $\mathfrak{I}$-lattice

In this section we try to put in an algebraic context all the conceptions of the $\mathfrak{I}$-lattice and develop it.

We begin with the definition of topological Boolean algebra (Rasiowa 1974).

Definition 5.1. A *topological Boolean algebra* is an abstract algebra $\langle B, 0, 1, \wedge, \vee, \sim, \bullet\rangle$ such that $\langle B, 0, 1, \wedge, \vee, \sim\rangle$ is a Boolean algebra and $\bullet$ is an unary operation that respects the following:

(i) $\bullet(a \vee b) = \bullet a \vee \bullet b$
(ii) $a \vee \bullet a = \bullet a$
(iii) $\bullet \bullet a = \bullet a$
(iv) $\bullet 0 = 0$.

Definition 5.2. An *$\mathfrak{I}$-lattice* is an abstract algebra $(B, 0, 1, \wedge, \vee, \sim, \bullet, \Cup)$ such that $\langle B, 0, 1, \wedge, \vee, \sim, \bullet\rangle$ is a topological Boolean algebra and $\Cup$ is a binary operation defined by:

$$a \Cup b = \bullet(a \vee b).$$

Theorem 5.3. *Every lattice is an $\mathfrak{I}$-lattice.*

Proof: Given a lattice $\mathfrak{L}$, if we take the identity function i as $\bullet$, naturally we have an $\mathfrak{I}$-lattice. ■

However we can prove that not every $\mathfrak{I}$-lattice is a lattice, but we shall postpone this and other proofs for another paper.

6 Conclusions

The algebraic aspect of structures that arises from motivations taken from quantum mechanics is of course very rich and can motivate not only mathematical developments but some other 'physical' ones, closer to some assumptions made by quantum theory itself. The case of indiscernibility of quantum objects is a field that has raised much discussion but mainly from a philosophical point of view. Quasi-set theory is the first mathematical theory that intents to consider seriously the issue of pursuing indiscernibility not as something that need to be treated by a trick (as by postulating symmetry conditions) but as something that would hold from the start. To develop the algebraic counterpart sounds quite natural, and in this paper we have started this job. We aim at to improve the results of this paper in further works.

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