Paraconsistent Logics and Paraconsistency

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Abstract

In this article, we provide a survey of several paraconsistent logics (PL) and some of the philosophical issues they raise. We focus especially on the various kinds of applications that these logics have had. In particular, we consider C-logics, including their semantic properties, and the theory of descriptions associated with them. We present various kinds of paraconsistent set theories, and discuss how they can be used to develop paraconsistent mathematical theories, including theories about Russell sets, Russell relations, and paraconsistent Boolean algebras. We then examine discussive logic and its application to the foundation of physical theories and to the formal representation of partial truth. We then go on to consider different axiomatizations of annotated logics and their use in fuzzy set theory. Finally, after discussing additional developments in PL, we conclude the article by examining different applications of PL in technology, informatics, foundations of physics, morality, and law.

Key words: paraconsistent logic, paraconsistency, contradiction, law of non-contradiction, ex falso quodlibet, explosion, C-logics, Jaskowski's logic, discussive logic, annotated logic, set theory, Russell set, paraconsistent set theory, fuzzy set theory, C-systems, partial truth, da Costa, Priest, Batens.

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1 Introduction

> "I predict a time when there will be mathematical investigations of calculi containing contradictions, and people will actually be proud of having emancipated themselves from contradictions."
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> L. Wittgenstein
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> "As a lightning clears the air of unpalatable vapors, so an incisive paradox frees the human intelligence from the lethargic influence of latent and unsuspected assumptions. Paradox is the slayer of Prejudice."
>
> J. J. Sylvester

IN A FEW WORDS, paraconsistent logics (PL) are the logics of inconsistent but non-trivial theories. A deductive theory is paraconsistent if its underlying logic is paraconsistent. A theory is inconsistent if there is a formula (a grammatically well-formed expression of its language) such that the formula and its negation are both theorems of the theory; otherwise, the theory is called consistent. A theory is trivial if all formulas of its language are theorems. Roughly speaking, in a trivial theory 'everything' (expressed in its language) can be proved. If the underlying logic of a theory is classical logic, or even any of the standard logical systems like intuitionistic logic, inconsistency entails triviality, and conversely. So, how can we speak of inconsistent but non-trivial theories? Of course, by changing the underlying logic to one which admits inconsistency without making the system trivial. Paraconsistent logics do just this job.

Our use of terms like 'consistency', 'inconsistency', 'contradictory' and similar ones is syntactical, which is in accordance with the original metamathematical terminology of Hilbert and his school. In order to treat such terms from a semantic point of view, in the field of paraconsistency, one must be able to build, first, a paraconsistent set theory. This is possible, as we will see, although most semantics for paraconsistent logics are classical, i.e., constructed inside classical set theories. So, to begin with, it is best to employ the above terms syntactically.

1.1 The origins

The origins of paraconsistent logics go back to the first systematic studies dealing with the possibility of rejecting or restricting the law (or principle) of non-contradiction, which (in one of its possible formulations) says that a formula and its negation cannot both be true. The law of non-contradiction is one of the basic laws of traditional, or classical (Aristotelian), logic. This principle is important. After all, since inconsistency entails triviality, an inconsistent set of premises yields any well-formed statement as a consequence. The result is that the set of consequences of an inconsistent theory, or set of premises, will explode into triviality and the theory is rendered useless.

Another way of expressing this fact is by saying that under classical logic the closure of any inconsistent set of sentences includes every sentence. It is this which lies behind Popper's famous statement that the acceptance of inconsistency ''… would mean the complete breakdown of science" and that an inconsistent system is ultimately uninformative.[1]

Inconsistencies appear in various levels of discussion of science and philosophy. For instance, Peirce's world of 'signs' (in which we inhabit) is an inconsistent and incomplete world. Bohr's theory of the atom is one of the well-known examples in science of an inconsistent theory. The old quantum theory of black-body radiation, Newtonian cosmology, the (early) theory of infinitesimals in the calculus, the Dirac δ-function, Stokes' analysis of pendulum motion, Michelson's 'single-ray' analysis of the Michelson-Morley interferometer arrangement, among others, can also be considered as cases of inconsistencies in science. Given cases such as these, it seems clear that we should not eliminate a priori inconsistent theories, but rather investigate them. In this context, paraconsistent logics acquire a fundamental role within science itself as well as in its philosophy. As we will see below, due to the wide range of applications which nowadays have been found for these logics, they have an important role in applied science as well.

The forerunners of paraconsistent logics are Jean Łukasiewicz and Nicolai I. Vasiliev. Independently of each other, both suggested in 1910 and 1911 that 'non-Aristotelian' logics could be obtained by rejecting the law of non-contradiction.[2] Although Łukasiewicz did not construct any system of paraconsistent logic, his ideas on the principle of non-contradiction in Aristotle influenced his student S. Jaśkowski in the construction of 'discussive' (or 'discursive') logic in 1948. (We will comment on Jaśkowski's systems below.) In 1911, 1912 and 1913, inspired by the works of Lobachewski on non-Euclidean geometry, initially called 'imaginary geometry', Vasiliev envisaged an *imaginary logic*, a non-Aristotelian logic where the principle of non-contradiction was not valid in general. According to Arruda, Vasiliev did not believe that there exist contradictions in the real world, but only in a possible world created by the human mind. Thus, he hypothesized imaginary worlds where the Aristotelian principles were not valid, even though Vasiliev did not develop his ideas in full.[3]

The very first logician to construct a formal system of paraconsistent logic was Stanislaw Jaśkowski in 1948. His motivations came from his interests in systematizing theories that contain contradictions, such as dialectics, as well as to study theories where contradictions are caused by vagueness. He was also interested in the study of empirical theories whose postulates include contradictory assumptions (see section 4). Despite the wide range of possible applications, Jaśkowski's discussive logic was restricted to the propositional level. In 1958, the first author of this paper, independently of Jaśkowski, began the general study of contradictory systems [65]. Ever since, da Costa has developed several systems related to paraconsistency (for instance, 'paraclassical logic' – see section 7.3), showing how to deal with inconsistencies from dif-

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[1] For further details and references, see [97], Chap. 5.

[2] For further historical details on PL, see [15], [16], [17], [46], [102], [124], [217], [142], [109].

[3] Arruda systematized some points on Vasiliev's imaginary logic, giving rise to three systems of paraconsistent logic [14]. Nowadays, the logical works of Vasiliev have been studied in Russia, specially by V. A. Bazhanov.

ferent perspectives. He apparently became the first logician to develop strong logical systems involving contradictions which could be useful for substantive parts of mathematics as well as the empirical and human sciences. It should be remarked that the adjective ’paraconsistent’ (which means something like ‘at the side of consistency’) was suggested by F. Miró-Quesada, in 1976, in a letter to da Costa.

Already in the sixties, the interest in logics dealing with inconsistencies began in other parts of the world as well, particularly in Poland, Australia, United States, Italy, Argentina, Belgium, Ecuador, and Peru, mainly for its relations to da Costa’s logics and to relevant and dialectical logic. Of course, in this paper, we cannot refer to all of these tendencies nor do justice to all the authors involved. For the historical details, we recommend the reading of the papers mentioned above.

At least two facts have contributed to emphasize the relevance of these developments. The first is that in 1990, *Mathematical Reviews* added a new entry, 03B53, called ‘Paraconsistent Logic’. From 2000 on, the title was changed to ‘Logics admitting inconsistency (paraconsistent logics, discussive logics, etc.)’, thus encompassing a wider subject. The second fact is that since 1996, several World Congresses on Paraconsistency have been organized.[4] Nowadays ’paraconsistency’ can be regarded as a field of knowledge. But perhaps the most surprising fact concerning paraconsistent logic is related to its applications. As we will note later, there have been applications not only to the foundations of science and its philosophical analysis, but even to technology. Here we do not have space to present all the details, but the references list the original sources.

1.2 On the nature of logic

If one wishes to understand the meaning and nature of logic, it is important to take into account that logic today is a field of knowledge of the same nature as mathematics. The results achieved in logic can be compared to those of mathematics and the empirical sciences in their depth and originality (let us just mention Gödel’s theorems, the results in recursion theory and in the theory of models). And similarly to mathematics, we can divide logic into two domains: pure logic and applied logic. ‘Pure’ logic, similarly to pure mathematics, can be developed in principle *in abstracto*, independently of possible applications. In particular, we can study paraconsistent logic or intuitionistic logic by themselves, basically with the aim of exploring their abstract mathematical properties. From this point of view, in developing a logical system, the logician can proceed as Hilbert suggested, when he said that “[t]he mathematician will have to take account not only of those theories that come near to reality, but also, as in geometry, of all logically possible theories” [147]. To sum up, from the pure viewpoint, logic studies certain abstract structures, such as formal languages, models, and Turing machines, independently of their applications.

Following Hilbert’s suggestion, we can develop abstract (pure) systems where some principle of classical logic is violated, for instance, the principle that entails that from contradictory premises any formula can be derived; in symbols, $\alpha \wedge \neg\alpha \vdash \beta$. (The corresponding law, $(\alpha \wedge \neg\alpha) \rightarrow \beta$, is Duns Scotus Law, valid not only in classical logic,

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[4] See `www.cle.unicamp.br/wcp3` for the page of the third congress.

but in almost all known logical systems, including intuitionistic logic.)[5] This is the way taken by Vasiliev in the construction of his imaginary logics.

But to develop a logic we can also proceed from the *applied* point of view. In this case, we look at some domain of knowledge where it seems that some logic (in particular, a paraconsistent one) could be used to describe certain abstract structures that reflect the way certain deductive inferences are made in that domain. One of the best known examples is provided by Birkhoff and von Neumann's approach to quantum logic, when they insisted that quantum mechanics would demand a logic distinct from the classical one, giving rise to a whole new field of investigation, quantum logic (see [120]).

It is sometimes claimed that non-classical logics need to be developed *because* classical logic is wrong, and so it must be replaced by a suitable logic, in accordance to some philosophical criterion (for a discussion, see [143]). This would be the case, for example, of the intuitionistic Brouwer-Heyting logic if we consider it as a culmination of Brouwer's original philosophy of mathematics. Brouwer's stance implies that, in a certain sense, classical mathematics has basic shortcomings and that a constructive mathematics should take its place; the underlying logic of this constructive mathematics is a new one, different from classical logic. In particular, it may be argued that, in domains involving inconsistency (if really there are any), some other logic should be used instead of classical logic. Nowadays, there are also philosophers who believe that, in these fields, classical logic should be replaced by another logic (most of them think that the right logic would be relevant).

But this is not our view regarding the rejection of classical logic. We think that classical logic is a key subject that has, and will continue to have, strong interest and applications. The only difference is that, in certain domains, other logics, in particular paraconsistent logics, may be more adequate to make explicit some of the underlying structures that (apparently) are being assumed in these domains. Classical logic can't do that in every domain. This does not show that classical logic is wrong, but that its area of application should be restricted. The use of non-classical logics in systematizing certain domains helps us understand important aspects of these domains. For example, with PL, the nature of negation has been better understood, and the significance of Russell set can be appreciated – see section 3.

Furthermore, as will become clear below, in our view, classical logic is valid in its particular domain of application. With regard to PL, two possibilities emerge. In certain contexts, PL can be viewed as a 'heterodox' logic, as a 'rival' logic [143], that is, as a logic that deviates from classical logic with respect to some of its principles. But, in other contexts, PL can be viewed as a *supplement* to classical logic. After all, certain paraconsistent logics coincide with classical logic if we take into consideration just what are called 'well-behaved propositions' (roughly, those propositions that obey the principle of non-contradiction). In short, we don't intend to play according to PL rules alone. PL may be useful in some domains, as shown below, but we will continue to use classical logic, or any other logic for that matter, whenever it is convenient or appropriate.

So, we are inclined to agree with Gonseth [133, Chap. 8] and sustain that (applied)

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[5] Some authors attribute this principle, also known as 'the principle of explosion', to Pseudo Scotus.

logic has an empirical counterpart. Nevertheless, we also have some reservations about Gonseth's picture. First, given our distinction between pure and applied logic, it is not necessary to eliminate a certain a priori aspect of logic, as Gonseth apparently wants (according to him, ''logic is the science of an arbitrary object"). This does not imply that we are endorsing the position that there is just one logic, which is independent of any domain of knowledge. As noted above, logic *can* be studied independently of any application, as a pure mathematical system, and in this sense, it can be considered as weakly a priori. So, even an applied logical system possesses an a priori dimension, in addition to its a posteriori one. In fact, we can begin by studying a logical system (say, some quantum logic) motivated by empirical considerations from science, but then proceed to verify whether this domain can be axiomatized, prove a completeness theorem for the resulting system, study other metalogical properties of the system, and so on.

From another point of view, however, logic deals with the underlying structures of inference of particular domains or theories. In this sense, a particular field (such as the quantum world, to continue with the example) may suggest that a different logic (that is, other than the classical one) is useful to accommodate certain features that cannot be dealt with by classical logic. For example, suppose we accept the view (advanced by E. Schrödinger, M. Born and others) that quantum objects are *non-individuals*, that is, they have no individuality since a quantum object is always indistinguishable of any other of a similar kind. It then seems that if we look at the quantum world as constituted by entities of this kind, classical logic (with Leibniz's Principle of the Identity of Indiscernibles) and classical mathematics (based on the notion of set, that is, collections of *distinguishable* objects) should be revised. This is particularly the case if we want to accommodate entities that, in certain contexts, can be regarded as 'individuals' (for instance, when an apparatus is prepared to work with particles), but in other contexts cannot be regarded as 'individuals' (say, when waves are taken into consideration). (Concerning these points, see [130].) So, different 'perspectives' of a domain of science may demand distinct logical tools, which puts us in a different framework than the classical one.

However, let us insist, the possibility of using non-standard systems does not necessarily entail that classical logic is wrong, or that domains like quantum theory *need* at the moment another logic. Physicists and other scientists probably will continue to use classical (informal) logic in the near future. But we should realize that other forms of logic may help us understand certain features of these domains, which are not easily accommodated by classical means – as the concepts of non-individuality and complementarity in the quantum domain show (see [130], [99], [100]).

We don't think there is just one 'true logic'. After all, distinct logical systems can be useful to describe different aspects of knowledge. (The same point can be made about distinct mathematical systems, and perhaps even about different physical physical systems.) In other words, we defend a form of *logical pluralism*. But our proposal is not relativist, since it's not the case that anything goes as far as applied logic is concerned. We can always rule out certain applied logical systems as being inadequate for certain domains. For instance, to capture constructive features of mathematical reasoning, classical logic is clearly inadequate; intuitionistic logic isn't. (We will return to the issue of logical pluralism below.) With regard to PL, we claim, with Granger

[139], that paraconsistent logic can, and should, be employed in the development of certain domains, but only as a preliminary tool. In the end, classical logic may eventually replace it as the underlying logic of these domains. Our position does not excludes Granger's.

In summary, there are in principle various 'pure' logics whose potential applications depend not only on a priori reasons, but, above all, on the nature of the applications one has in mind. This is also true of PL.

This paper is organized as follows. In the next section, we present da Costa's C-logics. We then turn to paraconsistent set theories, and show, in particular, how they accommodate inconsistent objects, such as Russell set. Next, we examine Jaśkowski's discussive logic, and show how it can be used in the formulation of the concept of partial truth. We then examine annotated logic, and some of its applications. Since all of these logics have been applied so extensively, such applications are here only touched upon (but references are given). Limitations of space also prevent us from examining all the related paraconsistent systems that have been developed in the literature. In fact, it would be impossible to do justice to all developments of PL and their corresponding applications. So, we limit ourselves to present the ideas and results with which our work is more closely related.

2 The C-logics

In this section, we study a class of logics termed C-logics, which show that it is possible to elaborate strong paraconsistent logical systems. In particular, within some of these systems, it is possible to build set theories and paraconsistent mathematics such that they contain standard mathematics. So, we can say that certain paraconsistent logico-mathematical systems increase the scope of traditional mathematics. In other words, it is possible to construct strong inconsistent systems without the immediate danger of trivialization. And there is no difficulty in reproducing within these systems the usual theories of logic and mathematics.

We begin with the propositional paraconsistent logic and, little by little, show how it is possible to get paraconsistent set theories and paraconsistent mathematics.

The way we approach the subject will be that of pure mathematics, in the same way as group theory or projective geometry of several dimensions are developed. The significance of all of this to the 'real world', that is, the possible applications of the resulting systems, will be discussed throughout the paper.

2.1 The propositional calculi C_n

Our initial goal is to develop propositional calculi that can be the basis of inconsistent but non-trivial theories.

We recall that a theory T, whose underlying logic is L and whose language is $\mathcal{L}$, is *inconsistent* if there is a formula α such that both α and $\neg\alpha$ (the negation of α) are theorems of T; otherwise, T is *consistent*. T is *trivial* if all formulas of $\mathcal{L}$ are theorems of T; otherwise, T is *non-trivial*. L is *paraconsistent* if it can be the underlying logic of inconsistent but non-trivial theories.

An expression of the form $\alpha \wedge \neg\alpha$, where $\wedge$ is the symbol for the conjunction, is called a *contradiction*. In general, if a theory is inconsistent, its logic enables us to derive, for whatever formula α, a contradiction $\alpha \wedge \neg\alpha$ from α and $\neg\alpha$; on the other hand, in most logics, from $\alpha \wedge \neg\alpha$ we can deduce both α and $\neg\alpha$. So, it is usual to call the trivial theories contradictory, which means that in such theories there are contradictions as theorems (or, equivalently, contradictory theorems).

We will begin by presenting the propositional calculus C_1. It seems natural that it should contain the usual connectives: $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), and $\neg$ (negation); equivalence ($\leftrightarrow$) is defined as usual (see Definition 2.1.1 below). Furthermore, it seems also natural that C_1 should be composed of most all of the valid schemes and rules of classical propositional calculus, obeying the following conditions:[6]

I. In C_1 it should be not generally valid the principle of non-contradiction.

II. From two contradictory propositions, that is, one being the negation of the another, it should not be possible to deduce any proposition whatever.

The language $\mathcal{L}$ of C_1 contains the following primitive symbols: (i) propositional variables: a denumerable (infinite) set of propositional variables (formulas that are not analyzed at the propositional level); (ii) connectives: $\rightarrow$, $\wedge$, $\vee$ and $\neg$; and (iii) parentheses.

Formulas are defined as follows: (i) any propositional variable is a formula; (ii) if α and β are formulas, then $(\alpha \rightarrow \beta)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ and $\neg\alpha$ are formulas; (iii) the only formulas are those obtained from the preceding conditions (i) and (ii).

To facilitate the reading, we will adopt some conventions: (a) the symbol $\rightarrow$ is stronger than the others; (b) $\wedge$ and $\vee$ are stronger than $\neg$; and (c) external parentheses can be dispensed with.

Definition 2.1.1 $\alpha \leftrightarrow \beta =_{\text{def}} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$

Now we list the postulates of C_1 (axiom schemes, axioms, and primitive inference rules). To begin with, we adopt the postulates of positive intuitionistic propositional logic:

1. $\alpha \rightarrow (\beta \rightarrow \alpha)$
2. $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma))$
3. $\alpha \wedge \beta \rightarrow \alpha$
4. $\alpha \wedge \beta \rightarrow \beta$
5. $\alpha \rightarrow (\beta \rightarrow \alpha \wedge \beta)$
6. $\alpha \rightarrow (\alpha \vee \beta)$

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[6] Until section 5, validity means syntactic validity: a formula is valid in a calculus if it has a proof in such calculus or if it is a theorem of this calculus.

7. $\beta \to (\alpha \vee \beta)$

8. $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$

9. $\alpha,\ \alpha \to \beta \,/\, \beta$

Let us now examine negation. We could think about adding to the postulates above the characteristic postulate of negation taken from the minimal calculus, namely,

$$(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha). \tag{1}$$

However, this would not be adequate. After all, from (1) and the postulates above, we can deduce the schema

$$\neg(\alpha \wedge \neg\alpha).$$

This stands for the principle of non-contradiction, which, by condition (I) above, should not be valid in C_1. Furthermore, in the minimal calculus, we can prove that the negation of any proposition can be derived from a contradiction, and this is not adequate either (see condition (II)). In the minimal calculus, we have (using (1))

$$\alpha, \neg\alpha, \beta \vdash \alpha \qquad \text{and} \qquad \alpha, \neg\alpha, \beta \vdash \neg\alpha;$$

hence

$$\alpha, \neg\alpha \vdash \neg\beta.$$

As a result, by applying twice the deduction theorem (which is a consequence of the postulates for implication of the minimal calculus), we have

$$\vdash \alpha \to (\neg\alpha \to \beta).$$

However, it seems interesting to accept (1), at least for those β such that

$$\neg(\beta \wedge \neg\beta).$$

This motivates us to adopt the following schema:

$$\neg(\beta \wedge \neg\beta) \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)).$$

We think it is convenient that every proposition be either *true* or *false*, at least in principle. (Nonetheless, we accept that there might be propositions that are true and whose negations are also true.) It then seems convenient to include, in addition, the following schema:

$$\alpha \vee \neg\alpha.$$

Now, $\neg(\alpha \wedge \neg\alpha)$ means that α satisfies the law of non-contradiction, that is, α is *well-behaved*. If this is not the case, that is, if $\alpha \wedge \neg\alpha$ holds, then α is *ill-behaved*. We then introduce the following definition:

Definition 2.1.2 $\alpha^o =_{\mathsf{def}} \neg(\alpha \wedge \neg\alpha)$.

Let us now consider the schema

$$\neg\neg\alpha \to \alpha. \tag{2}$$

We can reason heuristically as follows: if α is well-behaved, we can suppose that it obeys classical logic, and so (2) holds. If α is ill-behaved, then both α and $\neg\alpha$ are true, and by the postulates for implication, it follows that any proposition entails α; in particular, $\neg\neg\alpha$ entails α. Thus, (2) seems to be acceptable.

Finally, given what was said above, we also adopt postulates that entail that formulas built with well-behaved formulas are also well-behaved. That is:

$$\alpha^o \wedge \beta^o \to (\alpha \wedge \beta)^o, \quad \alpha^o \wedge \beta^o \to (\alpha \vee \beta)^o, \quad \alpha^o \wedge \beta^o \to (\alpha \to \beta)^o.$$

These new postulates can be written as follows:

$$\alpha^o \wedge \beta^o \to (\alpha \wedge \beta)^o \wedge (\alpha \vee \beta)^o \wedge (\alpha \to \beta)^o.$$

We can then deduce

$$\alpha^o \wedge \beta^o \to (\alpha \leftrightarrow \beta)^o.$$

Note that there is no necessity of assuming that

$$\alpha^o \to (\neg\alpha)^o,$$

for this can be proven from the postulates above.

So, we have the following list of postulates for C_1:

$(\to_1)$ $\alpha \to (\beta \to \alpha)$

$(\to_2)$ $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

$(\to_3)$ $\alpha,\ \alpha \to \beta \,/\, \beta$

$(\wedge_1)$ $\alpha \wedge \beta \to \alpha$

$(\wedge_2)$ $\alpha \wedge \beta \to \beta$

$(\wedge_3)$ $\alpha \to (\beta \to \alpha \wedge \beta)$

$(\vee_1)$ $\alpha \to (\alpha \vee \beta)$

$(\vee_2)$ $\beta \to (\alpha \vee \beta)$

$(\vee_3)$ $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$

$(\neg_1)$ $\beta^o \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$

$(\neg_2)$ $\alpha^o \wedge \beta^o \to (\alpha \wedge \beta)^o \wedge (\alpha \vee \beta)^o \wedge (\alpha \to \beta)^o$

$(\neg_3)$ $\alpha \vee \neg\alpha$

$(\neg_4)$ $\neg\neg\alpha \to \alpha.$

We will show shortly that C_1 has a bivalent semantics. But before doing that, let us study the main properties of C_1.

The concept of (formal) deduction of a formula from a set of formulas (i.e., in the standard notation, $\Gamma \vdash \alpha$) is defined in the usual way. In this case, we say that α is a syntactic consequence of the formulas in Γ. From now on, capital Greek letters stand for collections of formulas, while small Greek letters stand for formulas. We have, in C_1:

Theorem 2.1.1

(a) $\{\alpha\} \vdash \alpha$,

(b) $\Gamma \vdash \alpha$ entails $\Gamma \cup \Delta \vdash \alpha$,

(c) if $\Gamma \vdash \gamma$ for any $\gamma \in \Delta$ and $\Delta \vdash \alpha$, then $\Gamma \vdash \alpha$.

Proof: Immediate, from the standard definition of syntactic consequence ($\vdash$). ▮

Definition 2.1.3 *α is a theorem of C_1 iff $\vdash \alpha$.*

As usual, $\vdash \alpha$ means $\emptyset \vdash \alpha$. The symbols $\Rightarrow$ and $\Leftrightarrow$ are metalinguistic abbreviations of implication and bi-implication respectively.

Theorem 2.1.2 *In C_1:*

(a) (Deduction theorem) $\Gamma \cup \{\alpha\} \vdash \beta \Rightarrow \Gamma \vdash \alpha \rightarrow \beta$

(b) (Modus ponens) $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$

(c) $\{\alpha,\beta\} \vdash \alpha \wedge \beta$, $\{\alpha,\beta\} \vdash \alpha$, $\{\alpha,\beta\} \vdash \beta$

(d) $\{\alpha\} \vdash \alpha \vee \beta$, $\{\beta\} \vdash \alpha \vee \beta$

(e) (Proof by cases) $(\Gamma \cup \{\alpha\} \vdash \gamma$ and $\Gamma \cup \{\beta\}) \vdash \gamma \Rightarrow \Gamma \cup \{\alpha \vee \beta\} \vdash \gamma$

(f) (Double negation elimination) $\{\neg\neg\alpha\} \vdash \alpha$.

Proof: As in classical logic. The deduction of the paraconsistent *reductio ad absurdum* is made using the rules for implication and the schema $(\neg_1)$. ▮

The next theorem is important, since it shows that conditions (I) and (II) that C_1 should obey are indeed satisfied.

Theorem 2.1.3 *In C_1, the following schemas do not hold:*

1. $\neg\alpha \rightarrow (\alpha \rightarrow \beta)$
2. $\neg\alpha \rightarrow (\alpha \rightarrow \neg\beta)$
3. $\alpha \rightarrow (\neg\alpha \rightarrow \beta)$
4. $\alpha \rightarrow (\neg\alpha \rightarrow \neg\beta)$

5. $\alpha \wedge \neg\alpha \to \beta$

6. $\alpha \wedge \neg\alpha \to \neg\beta$

7. $(\alpha \to \beta) \to ((\alpha \to \beta) \to \neg\alpha)$

9. $\alpha \to \neg\neg\alpha$

10. $(\alpha \leftrightarrow \neg\alpha) \to \beta$

11. $(\alpha \leftrightarrow \neg\alpha) \to \neg\beta$

12. $\neg(\alpha \wedge \neg\alpha)$

13. $\alpha \wedge \neg\alpha \to \neg(\alpha \wedge \neg\alpha)$

Proof: The result is established using the tables below, with 1 and 2 as designated values:

α	β	$\alpha \to \beta$	$\alpha \wedge \beta$	$\alpha \vee \beta$
1	1	1	1	1
2	1	1	1	1
3	1	1	3	1
1	2	1	1	1
2	2	1	1	1
3	2	1	3	1
1	3	3	3	1
2	3	3	3	1
3	3	1	3	3

α	$\neg\alpha$
1	3
2	1
3	1

▮

We emphasize that, in C_1, the following principles – that play an important role in certain paradoxes, since they lead to trivialization – are *not* valid:

$$\neg\alpha \to (\alpha \to \beta),\ \neg\alpha \to (\alpha \to \neg\beta),\ \alpha \wedge \neg\alpha \to \beta,\ \alpha \wedge \neg\alpha \to \neg\beta.$$

Theorem 2.1.4 *In C_1, the following version of reductio ad absurdum holds:*

$$(\Gamma \vdash \beta^o, \Gamma \cup \{\alpha\} \vdash \beta, \Gamma \cup \{\alpha\} \vdash \neg\beta) \Rightarrow \Gamma \vdash \neg\alpha.$$

Proof: By the deduction theorem and $(\neg_1)$. ▮

Corolary 2.1.1 *In C_1, the following rules hold:*

$$(\Gamma \cup \{\alpha\} \vdash \beta^o, \Gamma \cup \{\alpha\} \vdash \beta, \Gamma \cup \{\alpha\} \vdash \neg\beta) \Rightarrow \Gamma \vdash \neg\alpha,$$

$$(\Gamma \cup \{\neg\alpha\} \vdash \beta^o, \Gamma \cup \{\neg\alpha\} \vdash \beta, \Gamma \cup \{\neg\alpha\} \vdash \neg\beta) \Rightarrow \Gamma \vdash \neg\alpha.$$

Proof: From the previous theorem and from the fact that in C_1 we have: $\vdash \alpha \vee \neg\alpha$ and $\vdash \neg\neg\alpha \to \alpha$. ▮

Theorem 2.1.5 *If we add to the axioms of C_1 the principle of non-contradiction, $\neg(\alpha \wedge \neg\alpha)$, as an additional postulate, we get the classical propositional calculus (CPC).*

Proof: By adding the principle of non-contradiction to C_1, we get $\vdash (\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$, and, hence, the postulate system for CPC presented by Kleene [159, p. 82]. ▮

From now on, to simplify the notation, we will eliminate the symbols { and } when we write deductions.

Theorem 2.1.6 *In C_1, we have:*

1. $\alpha \to \alpha$
2. $\alpha \leftrightarrow \alpha$
3. $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$
4. $\alpha \to (\beta \to \gamma) \vdash \beta \to (\alpha \to \gamma)$
5. $\alpha \to (\beta \to \gamma) \vdash \alpha \wedge \beta \to \gamma$
6. $\alpha \wedge \beta \to \gamma \vdash \alpha \to (\beta \to \gamma)$
7. $\alpha \to \beta \vdash (\beta \to \gamma) \to (\alpha \to \gamma)$
8. $\alpha \to \beta \vdash (\gamma \to \alpha) \to (\gamma \to \beta)$
9. $\alpha \to \beta \vdash \alpha \wedge \gamma \to \beta \wedge \gamma$
10. $\alpha \to \beta \vdash \gamma \wedge \alpha \to \gamma \wedge \beta$
11. $\alpha \to \beta \vdash \alpha \vee \gamma \to \beta \vee \gamma$
12. $\alpha \to \beta \vdash \gamma \vee \alpha \to \gamma \vee \beta$
13. $\alpha \leftrightarrow \beta \vdash \beta \leftrightarrow \alpha$
14. $\alpha \leftrightarrow \beta, \beta \leftrightarrow \gamma, \gamma \vdash \alpha \leftrightarrow \gamma$
15. $\vdash (\alpha \leftrightarrow \beta) \leftrightarrow (\beta \leftrightarrow \alpha)$
16. $\vdash (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$
17. $\vdash (\alpha \to (\alpha \to (\alpha \to \beta))) \to (\alpha \to (\alpha \to \beta))$
18. $\vdash \underbrace{(\alpha \to \ldots (\alpha \to \beta) \ldots)}_{n+1\ times} \to \underbrace{(\alpha \to \ldots (\alpha \to \beta) \ldots)}_{n\ times}$

Proof: Immediate consequences of the way C_1 was constructed. ▮

Theorem 2.1.7 *Propositional positive intuitionistic logic is contained in C_1.*

Proof: This intuitionistic logic is, in fact, characterized by the postulates $(\neg_1)$ to $(\vee_3)$. ▮

Theorem 2.1.8 *In C_1, we have:*

1. $\beta^o, \alpha \to \beta \vdash \neg\beta \to \neg\alpha$
2. $\beta^o, \alpha \to \neg\beta \vdash \beta \to \neg\alpha$
3. $\beta^o, \neg\alpha \to \beta \vdash \neg\beta \to \alpha$
4. $\beta^o, \neg\alpha \to \neg\beta \vdash \beta \to \alpha$
5. $(\alpha \to \neg\alpha) \to \neg\alpha$
6. $(\neg\alpha \to \alpha) \to \alpha$

Proof: We prove item 3. We have: $\beta^o, \neg\alpha \to \beta, \neg\alpha \vdash \beta^o$; $\beta^o, \neg\alpha \to \beta, \neg\beta, \neg\alpha \vdash \neg\beta$; $\beta^o, \neg\alpha \to \beta, \neg\beta, \neg\alpha \vdash \beta$. Hence, $\beta^o, \neg\alpha \to \beta, \neg\beta \vdash \neg\neg\alpha$, therefore $\beta^o, \neg\alpha \to \beta, \neg\beta \vdash \alpha$. So, $\beta^o, \neg\alpha \to \beta \vdash \neg\beta \to \alpha$. ▮

Remark: Since C_1 has several of the usual properties of classical propositional calculus, we can prove various other schemas as well, as is easy to see.

Theorem 2.1.9 *C_1 is a sub-calculus of the classical propositional calculus.*

Proof: All postulates of C_1 are valid in the classical propositional calculus. Furthermore, Theorem 2.1.3 shows that there are schemas that are valid in the latter calculus, but which are not valid in C_1. ▮

Theorem 2.1.10 *In C_1, the following schemas are not valid:*

$$(\alpha \wedge \beta) \wedge \neg\alpha \to \beta \quad \textit{and} \quad \alpha \vee \beta \to (\neg\alpha \to \beta).$$

Proof: If these schemas were valid in C_1, then using the deduction theorem, modus ponens and the schema $\alpha \to \alpha \vee \beta$, we could derive in C_1 the schema $\alpha \to (\neg\alpha \to \beta)$. ▮

The previous theorem could also be proven using the tables above. As a result, several other schemas that are valid in classical logic are not theorems of C_1.

Corolary 2.1.2 *The following rules are not valid in C_1 (disjunctive syllogisms):*

$$\frac{\alpha \vee \beta, \neg\alpha}{\beta} \quad \textit{and} \quad \frac{\neg\alpha \vee \beta, \alpha}{\beta}.$$

Theorem 2.1.11 *If $\alpha_1, \ldots, \alpha_n$ are the prime components (propositional variables) of a formula α, then a necessary and sufficient condition for α to be provable in the classical propositional calculus is that $\alpha_1^o, \ldots, \alpha_n^o \vdash \alpha$ in C_1.*

Proof: If $\alpha_1^o, \ldots, \alpha_n^o \vdash \alpha$ in C_1, then $\vdash \alpha$ in the classical propositional calculus, since β^o is an abbreviation of $\neg(\beta \wedge \neg\beta)$. Now, if $\vdash \alpha$ in the classical propositional calculus (supposed axiomatized as in [159]), then there exists a proof P of α in this calculus, in which only those formulas whose prime components are among $\alpha_1, \ldots, \alpha_n$ appear. So, if k is one of the formulas in P, given postulate $(\neg_2)$, we have that $\alpha_1^o, \ldots, \alpha_n^o \vdash k^o$ in C_1. Furthermore, in C_1, we have that $\alpha_1^o, \ldots, \alpha_n^o \vdash (\gamma \to \delta) \to ((\gamma \to \neg\delta) \to \neg\gamma)$, with the usual restrictions. But, since every postulate of the classical propositional calculus of [159] is valid in C_1, with the exception of $(\theta \to \pi) \to ((\theta \to \neg\pi) \to \neg\theta)$, we see that P can be transformed into a deduction, in C_1, of α from $\alpha_1^o, \ldots, \alpha_n^o$. ▮

Theorem 2.1.12 *If $\alpha_1, \ldots, \alpha_n$ are the prime components (propositional variables) of the formulas Γ and of a formula α, then a necessary and sufficient condition for $\Gamma \vdash \alpha$ in the classical propositional calculus is that $\Gamma, \alpha_1^o, \ldots, \alpha_n^o \vdash \alpha$ in C_1.*

Proof: Analogous to the previous one. ▮

Theorem 2.1.13 (A. I. Arruda) *In C_1, we have: $\vdash \alpha^{oo}$.*

Proof: $\vdash \alpha^{oo}$ means $\neg(\alpha^o \wedge \neg\alpha^o)$, that is, $\neg(\alpha^o \wedge \neg\neg(\alpha \wedge \neg\alpha))$. But $\alpha^o \wedge \neg\neg(\alpha \wedge \neg\alpha) \vdash \alpha^o$ and $\alpha^o \wedge \neg\neg(\alpha \wedge \neg\alpha) \vdash \alpha \wedge \neg\alpha$, hence $\vdash \neg(\alpha^o \wedge \neg\neg(\alpha \wedge \neg\alpha))$, that is, $\vdash \alpha^{oo}$. ▮

Theorem 2.1.14 *In C_1, we have: $\vdash \alpha^o \to (\neg\alpha)^o$.*

Proof: We have: $\alpha^o, \neg\alpha \wedge \neg\neg\alpha \vdash \alpha$ and $\alpha^o, \neg\alpha \wedge \neg\neg\alpha \vdash \neg\alpha$. Given that $\alpha^o, \neg\alpha \wedge \neg\neg\alpha \vdash \alpha^o$, it follows that $\alpha^o \vdash \neg(\neg\alpha \wedge \neg\neg\alpha)$, and hence, $\alpha^o \vdash (\neg\alpha)^o$. Thus, $\vdash \alpha^o \to (\neg\alpha)^o$. ▮

Definition 2.1.4 (Strong negation) $\neg^\star\alpha =_{\mathsf{def}} \neg\alpha \wedge \alpha^o$

Theorem 2.1.15 *In C_1, $\vdash (\alpha \to \beta) \to ((\alpha \to \neg^\star\beta) \to \neg^\star\alpha)$*

Proof: We have

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \alpha^o \vdash \neg^\star\alpha, \quad \text{and} \tag{1}$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \neg\alpha^o \vdash \alpha \wedge \neg\alpha. \quad \text{But}$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \alpha \wedge \neg\alpha \vdash \beta,$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \alpha \wedge \neg\alpha \vdash \neg\beta \quad \text{and}$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta \vdash \alpha^o. \quad \text{Therefore,}$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \neg\alpha^o \vdash \alpha,$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \neg\alpha^o \vdash \neg\alpha,$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \neg\alpha^o \vdash \alpha^o, \quad \text{and}$$

$$\alpha \to \beta,\ \alpha \to \neg^\star\beta,\ \neg\alpha^o \vdash \neg^\star\alpha. \tag{2}$$

From (1) and (2), using proof by cases and excluded middle, we have that

$\alpha \to \beta,\ \alpha \to \neg^\star\beta \vdash \neg^\star\alpha$, hence

$\vdash (\alpha \to \beta) \to ((\alpha \to \neg^\star\beta) \to \neg^\star\alpha)$. $\blacksquare$

Theorem 2.1.16 $\vdash \alpha \to (\neg^\star\alpha \to \beta)$

Proof: We have:

$\alpha,\ \neg\alpha \wedge \alpha^o,\ \neg\beta \vdash \alpha$,

$\alpha,\ \neg\alpha \wedge \alpha^o,\ \neg\beta \vdash \neg\alpha$,

$\alpha,\ \neg\alpha \wedge \alpha^o,\ \neg\beta \vdash \alpha^o$. Hence,

$\alpha,\ \neg\alpha \wedge \alpha^o \vdash \neg\neg\beta$,

$\alpha,\ \neg\alpha \wedge \alpha^o \vdash \beta$ and $\alpha,\ \neg^\star\alpha \vdash \beta$. So,

$\vdash \alpha \to (\neg^\star\alpha \to \beta)$. $\blacksquare$

Theorem 2.1.17 $\vdash \alpha \vee \neg^\star\alpha$

Proof: We have:

$\vdash (\alpha \vee \neg^\star\alpha) \leftrightarrow \alpha \vee (\neg\alpha \wedge \alpha^o)$,

$\vdash (\alpha \vee \neg^\star\alpha) \leftrightarrow (\alpha \vee \neg\alpha) \wedge (\alpha \vee \alpha^o)$, and

$\vdash (\alpha \vee \neg^\star\alpha) \leftrightarrow (\alpha \vee \alpha^o)$.

But i) $\alpha^o \vdash \alpha \vee \alpha^o$ and ii) $\neg\alpha^o \vdash \alpha \wedge \neg\alpha \vdash \alpha \vdash \alpha \vee \alpha^o$. So, from i) and ii) it follows that $\vdash \alpha \vee \alpha^o$, hence $\vdash \alpha \vee \neg^\star\alpha$. $\blacksquare$

Theorem 2.1.18 *The connectives* $\to$, $\wedge$, $\vee$ *and* $\neg^\star$, *in* C_1, *satisfy all schemas and inference rules of classical propositional calculus.*

Proof: In C_1, the following schemas are valid:

$(\alpha \to \beta) \to ((\alpha \to \neg^\star\beta) \to \neg^\star\alpha)$,

$\alpha \to (\neg^\star\alpha \to \beta)$ and

$\alpha \vee \neg^\star\alpha$.

If we add to these schemas the postulates ($\to_1$) to ($\vee_3$) above, we get the postulates for the classical propositional calculus (see [159]). $\blacksquare$

Theorem 2.1.19 *In* C_1, *Peirce's Law is valid; that is,*

$$\vdash ((\alpha \to \beta) \to \alpha) \to \alpha.$$

Proof: Peirce's Law can be derived in the classical propositional calculus. So, by the previous theorem, it is provable in C_1. $\blacksquare$

Theorem 2.1.20 *In C_1, we have:* $\vdash \alpha \vee (\alpha \to \beta)$.

Proof: $\vdash \alpha \vee (\alpha \to \beta)$ is valid in the classical propositional calculus. ▮

The results above show that classical propositional calculus is, in a certain sense, contained in C_1, although C_1 is a sub-calculus of it.

Definition 2.1.5 *A schema or formula α* trivializes *a calculus C if by adding α to C, the new calculus is trivial, that is, all of its formulas are theorems.*

Theorem 2.1.21 *Any formula of the form $\alpha \wedge \neg^\star\alpha$ trivializes C_1.*

Proof: By the results above, we see that $\neg^\star$ is a classical negation. So, $\alpha \wedge \neg^\star\alpha$ entails any formula, that is, $\vdash \alpha \wedge \neg^\star\alpha \to \beta$. ▮

Theorem 2.1.22 *The schema $\alpha \leftrightarrow (\alpha \to \beta)$, where β is any formula, trivializes C_1.*

Proof: In C_1, we have (i) $\vdash \alpha \vee (\alpha \to \beta)$. Hence, from α and $\alpha \leftrightarrow (\alpha \to \beta)$, we deduce β. From $\alpha \to \beta$ and from $\alpha \leftrightarrow (\alpha \to \beta)$, we deduce β again. So, from (i), $\vdash \beta$. ▮

Since, in C_1, we have that: $\vdash \alpha \vee \underbrace{(\alpha \to (\alpha \to \ldots (\alpha \to \beta)\ldots)}_{n \text{ occurences of } \alpha}$, then the schema

$$\alpha \leftrightarrow \alpha \vee (\alpha \to (\alpha \to \ldots (\alpha \to \beta)\ldots),$$

where β is any formula, also trivializes C_1.

Theorem 2.1.23 *In C_1,*

$$\nvdash (\alpha \leftrightarrow (\alpha \vee \alpha)) \leftrightarrow (\neg\alpha \leftrightarrow (\neg(\alpha \vee \alpha)).$$

Proof: Use the table given at Theorem 2.1.3, and assign to α the value 2. ▮

Theorem 2.1.24 *In C_1, the following schema and rule are not valid:*

$$(\alpha \leftrightarrow \beta) \to ((\neg\alpha \leftrightarrow \neg\beta), \quad \frac{\alpha \leftrightarrow \beta}{\neg\alpha \leftrightarrow \neg\beta}.$$

Proof: Immediate consequence of the previous theorem. ▮

2.2 The hierarchy C_n, $0 \leq n \leq \omega$

The calculus C_1 is not the only that satisfies the conditions I and II formulated above (see page 7). Among other possible solutions, we will indicate, in what follows, a hierarchy of calculi which satisfy such conditions, except for the first one, which will be taken to be the classical propositional calculus. The hierarchy is the following:

$$C_0, C_1, C_2, \ldots, C_n, \ldots, C_\omega, \tag{3}$$

where C_0 is the classical propositional calculus and the remaining ones are defined below.

To begin with, let us introduce the following definition:

Definition 2.2.1

(i) $\alpha^{(1)}$ stands for α^{o}

(ii) $\alpha^{(n)}$ stands for $\alpha^{n-1} \wedge (\alpha^{(n-1)})^{o}$, $2 \leq n \leq \omega$.

The calculus C_n $(0 < n < \omega)$ is then individualized by the postulates $(\rightarrow_1)$ to $(\vee_3)$ above, plus $\alpha \vee \neg\alpha$, $\neg\neg\alpha \rightarrow \alpha$ and the following ones:

(n_1) $\beta^{(n)} \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha)$

(n_2) $\alpha^{(n)} \wedge \beta^{(n)} \rightarrow ((\alpha \rightarrow \beta)^{(n)} \wedge (\alpha \wedge \beta)^{(n)} \wedge (\alpha \vee \beta)^{(n)})$.

The calculus C_ω has as postulates $(\rightarrow_1)$ to $(\vee_3)$ plus $\alpha \vee \neg\alpha$ and $\neg\neg\alpha \rightarrow \alpha$.

We can see that, in $C_n, 1 \leq n \leq \omega$, we can substitute the schema $\beta^{(n)} \wedge \beta \wedge \neg\beta \rightarrow \gamma$ for the postulate (n_1). It is also easy to verify that the results above, proven for C_1, with suitable adaptations, can be also proven for $C_n, n = 2, 3, \ldots$

Theorem 2.2.1 *Each calculi in the hierarchy (3) is strictly stronger than those which follow it.*

Proof: (A. I. Arruda) The tables of the proof of Theorem 2.1.3, which we call here T_1, show that C_0 properly contains C_1. With the aim of proving that C_i properly contains C_{i+1}, $i = 1, 2, \ldots$, we define the following tables $T_2, T_3, \ldots$ (note that they do not include the tables for negation, which will be defined separately). T_2 is obtained from T_1 (as in the proof of Theorem 2.1.3) by the addition of a new value, 4, which will be the only non-designated value; we obtain T_n from T_{n-1} by adding a new value $n+2$, which will be the only non-designated one etc. The tables T_n are then obtained from T_{n-1}, $n = 2, 3, \ldots$, as follows. (Note that the rules below refer to the new arrangements of values, which result from the addition of a new value, and no changes are made, in the new tables, of those already obtained in the tables of order $n-1$.)

(1) *Conjunction*: If the components have different values, the conjunction will have the greatest value among the values of its conjuncts; if the values are equal, this will be the value of the conjunction.

(2) *Disjunction*: If the values of the components are distinct, the disjunction will have the smaller value among the values of the components; if they are equal, this value will be the value of the disjunction.

(3) *Implication*: If the values of the components are distinct, the implication will have the value of the consequent; if they are equal, the implication will have the value 1.

As for negation, T_n gives us the following table:

α	1	2	3	$\cdots$	n	n+1	n+2
$\neg\alpha$	n+2	1	2	$\cdots$	n-1	n	1

In the table above, $n = 1, 2, \ldots, n, n+1$ are the designated values and $n+2$ is the only non-designated one.

The tables T_n show that C_{n-1} strictly contains C_n. For instance, the postulate $\beta^{(n-1)} \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$ and the schema $\alpha^{(n-1)} \to (\neg\alpha)^{(n-1)}$ are not valid in C_n, which may be verified with some work ($n > 1$). Furthermore, C_ω is strictly weaker than all the other calculi of the hierarchy. ▮

2.3 Theories

In this section, we will make explicit reference to the calculus C_1, although the exposition can be extended to all the calculi $C_n, 0 \leq n \leq \omega$.

Definition 2.3.1 $\overline{\Delta} =_{\text{def}} \{\alpha : \Delta \vdash \alpha\}$

Theorem 2.3.1

(i) $\Delta \subset \overline{\Delta}$

(ii) $\Delta \subset \Gamma \Rightarrow \overline{\Delta} \subset \overline{\Gamma}$

(iii) $\overline{\overline{\Delta}} \subset \overline{\Delta}$

Proof: Immediate. ▮

Definition 2.3.2 *Δ is a theory if and only if $\overline{\Delta} = \Delta$.*

Definition 2.3.3 *Let Υ be the set of all formulas of C_1. Then Δ is inconsistent if there exists α such that $\Delta \vdash \alpha$ and $\Delta \vdash \neg\alpha$, otherwise Δ is consistent. Γ is trivial if $\overline{\Gamma} = \Upsilon$; otherwise, Γ is non-trivial. An inconsistent set is also called contradictory. A theory Δ is paraconsistent if it is inconsistent but non-trivial.*

Definition 2.3.4 *Expressions of either the form $\alpha \wedge \neg\alpha$ or $\neg\alpha \wedge \alpha$ are called contradictions.*

It is easy to see that Δ is inconsistent if and only if $\Delta \vdash \alpha \wedge \neg\alpha$ for some formula α.

Theorem 2.3.2 *C_1 can be the underlying calculus of paraconsistent theories.*

Proof: It suffices to consider the theory $\overline{\{\alpha \wedge \neg\alpha\}}$, where α is a propositional variable, and to apply the tables of Theorem 2.1.3. ▮

Definition 2.3.5 *A formula α trivializes a calculus C if, by adding α to C as a new axiom, the new resulting system is trivial. In this case, C is said to be finitely trivializable.*

For instance, the intuitionistic or classical implicative propositional calculi and the classical positive propositional calculus are not finitely trivializable, while the classical predicate calculus is.

Theorem 2.3.3 *A formula of the form $\alpha \wedge \neg\alpha \wedge \alpha^{(n)}$ trivializes $C_n, 1 \leq n < \omega$.*

Proof: Immediate. ▮

Theorem 2.3.4 *C_ω is not finitely trivializable.*

Proof: Let us consider the following matrix M, where 1 is designated:

α	β	$\alpha \to \beta$	$\alpha \wedge \beta$	$\alpha \vee \beta$
0	0	1	0	0
1	0	0	0	1
0	1	1	0	1
1	1	1	1	1

If α does not begin with the symbol $\neg$, then:

1) If the value of α is 1, then the value of $\neg^n\alpha$ is 1, where $\neg^n$ stands for $\neg\neg\ldots\neg$, ($\neg$ repeated n times, $n \geq 1$).

2) If the value of α is 0, then the value of $\neg^{2k}\alpha$ is 0, and the value of $\neg^{2k+1}\alpha$ is 1, for all $k = 0, 1, 2, \ldots$

M is sound for C_ω, as is easy to see. By induction on the length of the formulas, we can show that no formula assumes the value 0. So, there is no formula γ such that $\gamma \vdash \alpha$ (or $\vdash \gamma \to \alpha$), for any formula α of C_ω. ▮

The following results can also be proven without difficulty:

Theorem 2.3.5 *The calculi C_n, $0 < n < \omega$, with a finite number of propositional variables, are trivializable by formulas of the form $\alpha \wedge \neg\alpha \wedge \alpha^{(n)}$.*

Theorem 2.3.6 *C_ω, with a finite number of propositional variables, cannot be finitely trivializable.*

It should be remarked that if a theory is based on C_n, there is more risk of trivialization than if it were based on C_{n+1}. To obtain the ‘maximum security’ in avoiding trivialization, we should adopt C_ω. But the further we go in the hierarchy (3), the weaker the calculi we get.

Theorem 2.3.7 (A. I. Arruda) *In C_n, $1 \leq n \leq \omega$, there is no reduction of negations; that is, expressions like $\neg^p\alpha \leftrightarrow \neg^q\alpha$, for $p \neq q$ ($p, q = 0, 1, 2, \ldots$), are not valid in these calculi.*

Proof: It suffices to use the following truth-tables: the values of the tables are the integers ≥ 1 and the only non-designated value is 1. If d stands for designated value and $v(\alpha)$ is the value of α, then the tables for $\to$, $\wedge$, $\vee$ and $\neg$ are as follows:

$\to$: $v(\alpha \to \beta) = 1$ iff $v(\alpha) = d$ and $v(\beta) = 1$; $v(\alpha \to \beta) = 2$ otherwise.

$\wedge$: $v(\alpha \wedge \beta) = 2$ iff $v(\alpha) = d$ and $v(\beta) = d$, and $v(\alpha \wedge \beta) = 1$ otherwise.

$\vee$: $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$, and $v(\alpha \vee \beta) = 2$ otherwise.

$\neg$: $v(\neg\alpha) = 2$ iff $v(\alpha) = 1$ and $v(\neg\alpha) = 1$ otherwise. ▮

Corolary 2.3.1 (A. I. Arruda) *The calculi C_n, $1 \leq n \leq \omega$, do not have finite characteristic truth-tables; that is, finite tables such that a necessary and sufficient condition for α to be a theorem is that it assumes only designated values in these tables.*

Proof: From the hypothesis of the corollary, it is easy to see that if the calculi had characteristic finite tables, they would enable reduction of negations. ▮

2.4 Quantification

Corresponding to the hierarchy (3), we construct the corresponding first-order predicate calculi. These new calculi will be denoted by

$$C_0^\star, C_1^\star, C_2^\star, \ldots, C_n^\star, \ldots, C_\omega^\star. \tag{4}$$

To begin with, we will construct the calculus $C_1^\star$. Let $\mathcal{L}^\star$ be a first-order language that contains the usual symbols, connectives and quantifiers, as well as a denumerable family of individual variables, predicate symbols (one for each arity), and auxiliary symbols. We could also suppose that $\mathcal{L}^\star$ contains functional symbols, and in particular, individual constants; but we will not make this supposition here, except when explicitly mentioned. The notions of formula, free and bound variables in a formula, sentence (formula without free variables) etc. are standard. The notations and metalogical conventions extend those made for the propositional calculi.

$C_0^\star$ is the classical first-order predicate calculus. The postulates of $C_1^\star$ are those of C_1 (suitably adapted) plus the following:

(I) $\frac{\alpha \to \beta(x)}{\alpha \to \forall x \beta(x)}$

(II) $\forall x \alpha(x) \to \alpha(y)$

(III) $\alpha(x) \to \exists x \alpha(x)$

(IV) $\frac{\alpha(x) \to \beta}{\exists x \alpha(x) \to \beta}$

(V) $\forall x(\alpha(x))^o \to (\forall x \alpha(x))^o$

(VI) $\forall x((\alpha(x))^o \to (\exists x \alpha(x))^o$

where the variables x and y and the formulas α and β satisfy the usual restrictions.

Furthermore, an additional postulate needs to be adopted. To present it, let us say that two formulas are *congruent* if one can be obtained from the other by replacing bound variables or by suppressing vacuous quantifications (without confusion of variables). Here is the additional postulate:

(VII) $A \leftrightarrow B$, where A and B are congruent formulas.

A reason to accept postulates I - IV derives from the fact that they hold in classical logic and are acceptable intuitively. Postulates V and VI are assumed for reasons similar to those that motivate the postulates of C_1. Finally, VII is required for it seems correct to say that two congruent formulas are equivalent, and this fact cannot be deduced from the remaining postulates.

We can then establish the following theorem, whose proof is done in the same way as in classical logic.

Theorem 2.4.1 *All derived rules of theorem 2.1.2 hold in* $C_1^\star$. *Furthermore, we have:*

$$\alpha(x) \vdash \forall x\alpha(x) \qquad \forall x\alpha(x) \vdash \alpha(y)$$

$$\alpha(t) \vdash \exists x\alpha(x) \qquad \text{If } \Gamma \cup \{\alpha(x)\} \vdash \beta, \text{ then } \Gamma \cup \{\exists x\alpha(x)\} \vdash \beta,$$

with the usual restrictions (*see* [159]).

Note that in $C_1^\star$, similarly to what happens in the classical calculus $C_0^\star$, the above rules impose restrictions on the deduction theorem (see [159]).

The proofs of the following theorems are also obtained without difficulty.

Theorem 2.4.2 *In* $C_1^\star$*:*

$$\vdash \forall x(\alpha(x) \to \beta(x)) \to (\forall\alpha(x) \to \forall x\beta(x))$$

$$\vdash \forall x(\alpha(x) \to \beta(x)) \to (\exists\alpha(x) \to \exists x\beta(x))$$

Theorem 2.4.3 *Suppose that* $\gamma_1, \ldots, \gamma_n$ *are the quantificationally prime components of the formula* α *and the set of formulas* Γ. *In this case, if* $\Gamma \vdash \alpha$ *in the classical predicate calculus, then* $\{\gamma_1^{\mathrm{o}}, \ldots, \gamma_n^{\mathrm{o}}\} \cup \Gamma \vdash \alpha$ *in* $C_1^\star$*, and conversely.*

Corolary 2.4.1 *Let* $\alpha_1, \ldots, \alpha_n$ *be the quantificationally prime components of the formula* α. *Then a necessary and sufficient condition for* $\vdash \alpha$ *in* $C_0^\star$ *is that* $\{\alpha_1^{\mathrm{o}}, \ldots, \alpha_n^{\mathrm{o}}\} \vdash \alpha$ *in* $C_1^\star$.

Theorem 2.4.4 $C_0^\star$ *is included in* $C_1^\star$.

Proof: In the case of formulas whose prime components are well-behaved, Theorem 2.4.3 shows that, for these formulas, the laws and rules of $C_0^\star$ hold. But, from Theorem 2.1.18, C_1 is, in a certain sense, included in $C_1^\star$, and so is $C_0^\star$. ▮

Corolary 2.4.2 $C_1^\star$ *is undecidable.*

Proof: Given that $C_0^\star$ is included in $C_1^\star$, and $C_0^\star$ is undecidable (Church's theorem), so is $C_1^\star$. ▮

Theorem 2.4.5 *The following schemas are valid in* $C_1^\star$*, with the usual restrictions* (see [159, p. 162])*, and where* γ *is a formula in which* x *does not appear free:*

1. $\forall x\,\gamma \leftrightarrow \gamma$
2. $\exists x\,\gamma \leftrightarrow \gamma$
3. $\forall x\forall y\alpha(x,y) \leftrightarrow \forall y\forall x\alpha(x,y)$

4. $\exists x \exists y \alpha(x,y) \leftrightarrow \exists y \exists x \alpha(x,y)$

5. $\forall x \forall y \alpha(x,y) \to \forall x \alpha(x,x)$

6. $\exists x \alpha(x,x) \to \exists x \exists y \alpha(x,y)$

7. $\forall x \alpha(x) \to \exists x \alpha(x)$

8. $\exists x \forall y \alpha(x,y) \to \forall y \exists x \alpha(x,y)$

9. $\forall x(\alpha(x) \wedge \beta(x)) \leftrightarrow \forall x \alpha(x) \wedge \forall x \beta(x)$

10. $\exists x(\alpha(x) \vee \beta(x)) \leftrightarrow \exists x \alpha(x) \vee \exists x \beta(x)$

11. $\alpha \wedge \forall x \beta(x) \leftrightarrow \forall x(\alpha \wedge \beta(x))$

12. $\alpha \vee \exists x \beta(x) \leftrightarrow \exists x(\alpha \vee \beta(x))$

13. $\alpha \wedge \exists x \beta(x) \leftrightarrow \exists x(\alpha \wedge \beta(x))$

14. $\alpha \vee \forall x \beta(x) \to \forall x(\alpha \vee \beta(x))$

15. $\exists x(\alpha(x) \wedge \beta(x)) \to \exists x \alpha(x) \wedge \exists x \beta(x)$

16. $\forall x \alpha(x) \vee \forall x \beta(x) \to \forall x(\alpha(x) \vee \beta(x))$

Proof: Similar to the proofs in the classical calculus (the same point also applies to some other schemas that hold in $C_1^\star$). ▮

Theorem 2.4.6 *In* $C_1^\star$,

$\forall x(\alpha(x))^o \vdash \exists x \alpha(x) \leftrightarrow \neg\forall \neg\alpha(x)$

$\forall x(\alpha(x))^o \vdash \forall x \alpha(x) \leftrightarrow \neg\exists x \neg\alpha(x)$

$\forall x(\alpha(x))^o \vdash \neg\forall x \alpha(x) \leftrightarrow \exists x \neg\alpha(x)$

$\forall x(\alpha(x))^o \vdash \neg\exists x \alpha(x) \leftrightarrow \forall x \neg\alpha(x)$

We define now the *k-transform* of a formula α, where k is a numeral among $\underline{1}, \underline{2}, \ldots$, that is, constant symbols in correspondence with the natural numbers $1, 2, \ldots$ [159, pp. 177ff]. By hypothesis, these symbols do not belong to the language of $C_0^\star$ and of $C_1^\star$.

Case 1: If α has no free variables, its k-transform is obtained as follows: each part of α of the form $\forall x \beta(x)$ or of the form $\exists x \beta(x)$ is replaced respectively by $\beta(\underline{1}) \wedge \beta(\underline{2}) \wedge \ldots \wedge \beta(\underline{n})$ or by $\beta(\underline{1}) \vee \beta(\underline{2}) \vee \ldots \vee \beta(\underline{n})$. Hence, this k-transform has no (individual) variables.

Case 2: If α has free variables, that is, if α is $\alpha(x_1, \ldots, x_m)$, then its k-transforms are obtained as follows: (a) we replace the variables $x_1, \ldots, x_m$ by permutations of $1, 2, \ldots, k$ with repetitions of order k; (b) we then take the k-transforms of the resulting formulas of (a).

Theorem 2.4.7 *If* $\Gamma \vdash \alpha$ *in* $C_1^\star$*, then any k-transform of* α *is deducible, in* C_1*, from the k-transforms of the formulas of* Γ*.*

Proof: By induction on the length of a deduction of α from Γ in the calculus $C_1^\star$. The proof is similar to the classical case (that is, similar to $C_0^\star$). ▮

Corolary 2.4.3 *If* $\vdash \alpha$ *in* $C_1^\star$*, then the k-transforms of* α *are provable in* C_1*.*

Corolary 2.4.4 *If the formula* α *has only predicate symbols of arity zero, that is, with zero associated variables, and* $\vdash \alpha$ *in* $C_1^\star$*, then* $\vdash \alpha$ *in* C_1*.*

Proof: It suffices to note that, in this case, the only *k*-transform of α is α itself. ▮

Remark: Corollary 2.4.4 is important for it emphasizes that those propositional schemas that do not hold in C_1 continue not to hold in $C_1^\star$. In other words, if we add to C_1 the postulates and specific formation rules of $C_1^\star$, no new result is obtained relatively to the *pure* formulas of C_1. Furthermore, Theorem 2.4.7 can be extended to sub-systems of the classical propositional calculus (and of intuitionistic propositional calculus) and their corresponding quantification theories. This is the case, for instance, of classical positive logic (and intuitionistic positive logic) as well as the propositional intuitionistic calculus and the minimal intuitionistic calculus. Theorem 2.4.7 applies to the classical calculi C_0 and $C_0^\star$ and constitutes the so-called Hilbert-Bernays Theorem on *k*-transforms.

Theorem 2.4.8 *In* $C_1^\star$*, the following schemas are not valid:*

1. $\neg\exists x\neg\alpha(x) \leftrightarrow \forall x\alpha(x)$
2. $\neg\forall x\neg\alpha(x) \leftrightarrow \exists x\alpha(x)$
3. $\neg\exists x\alpha(x) \leftrightarrow \forall x\neg\alpha(x)$
4. $\exists x\neg\alpha(x) \leftrightarrow \neg\forall x\alpha(x)$

Proof: It is easy to see that the 2-transforms of these schemas are not provable in $C_1^\star$. It suffices to take the 2-transforms and to use the tables of Theorem 2.1.3. ▮

Theorem 2.4.9 $C_1^\star$ *is strictly weaker than* $C_0^\star$*, and* $C_0^\star$ *can be obtained from* $C_1^\star$ *by adding to its postulates the following schema:* $\neg(\alpha \wedge \neg\alpha)$*.*

Proof: The result is a consequence of the fact that C_0 is obtained from C_1 if we add $\neg(\alpha \wedge \neg\alpha)$ to C_1 as a postulate. ▮

The remaining calculi $C_n^\star$, $2 \leq n < \omega$ of the hierarchy (4) are obtained by adding to C_n, with obvious adaptations, the postulates I-IV and VII above, plus the following ones:

(V_n) $\forall\, (\alpha(x))^{(n)} \rightarrow (\forall x\alpha(x))^{(n)}$

(VI_n) $\forall\, (\alpha(x))^{(n)} \rightarrow (\exists x\alpha(x))^{(n)}$

$C^\star_\omega$ is obtained from C_n by adding to its postulates the schemes IV and VII.

It is clear that the calculi $C^\star_n$, $2 \leq n < \omega$ have properties similar to those of $C^\star_1$. In particular, the following results hold.

Theorem 2.4.10 *Let the quantificationally prime components of the formula α and of the formulas of Γ be $\alpha_1, \ldots, \alpha_m$. In this case, if $\Gamma \vdash \alpha$ in $C^\star_0$, then $\alpha_1^{(n)}, \ldots \alpha_m^{(n)}, \Gamma \vdash \alpha$ in $C^\star_n$, $1 \leq n < \omega$, and conversely.*

Theorem 2.4.11 *$C^\star_n$, $0 \leq n < \omega$, is undecidable.*

Theorem 2.4.12 *(Essenin-Volpin) $C^\star_\omega$ is undecidable.*

Theorem 2.4.13 *If $\Gamma \vdash \alpha$ in $C^\star_n$, $0 \leq n \leq \omega$, then the k-transforms of α are deducible, in C_n, from the k-transforms of the formulas of Γ.*

Theorem 2.4.14 *If $\vdash \alpha$ in $C^\star_n$, $0 \leq n \leq \omega$, then the k-transforms of α are provable in C_n.*

Theorem 2.4.15 *If formula α belongs to C_n, $0 \leq n < \omega$, and $\vdash \alpha$ in $C^\star_n$, then $\vdash \alpha$ in C_n.*

Theorem 2.4.16 *Every calculus of the hierarchy $C^\star_n$, $1 \leq n \leq \omega$, is a proper subsystem of $C^\star_0$.*

We can prove for the hierarchy (4) results analogous to those established for the corresponding propositional calculi. For instance, we have:

Theorem 2.4.17 *Every calculus of the hierarchy $C^\star_n$, $0 \leq n \leq \omega$, is strictly stronger than those that follow it.*

Theorem 2.4.18 *Every calculus $C^\star_n$, $0 \leq n < \omega$, is finitely trivializable.*

Theorem 2.4.19 *$C^\star_\omega$ is not finitely trivializable.*

2.5 Equality

From the hierarchy (4), we can construct the following hierarchy of predicate calculi with equality:

$$C^=_0, C^=_1, C^=_2, \ldots, C^=_n, \ldots, C^=_\omega. \tag{5}$$

This is done by adding to their languages the binary predicate symbol of equality, =, with suitable modifications in the concept of formula, and by adding the following two postulates:

(I') $\forall x(x = x)$

(II') $x = y \to (\alpha(x) \to \alpha(y))$

with the same restrictions adopted in the classical calculus. Here, $C^=_0$ stands for the classical first-order predicate calculus with equality. We begin by studying $C^=_1$.

Theorem 2.5.1 *We have in* $C_1^=$*:*

1. $\vdash x = x$

2. $\vdash x = y \to y = x$

3. $\vdash x = y \wedge y = z \to x = z$

Theorem 2.5.2 *We have in* $C_1^=$*:*

1. $\vdash x = y \leftrightarrow (\alpha(x) \to \alpha(y))$

2. $\forall t(\alpha(t))^o \vdash \alpha(x) \wedge \neg\alpha(y) \to x \neq y$, where $x \neq y$ stands for $\neg(x = y)$.

Proof: The first schema is proven in the same way as in $C_0^=$. As for the second, note that $\forall t(\alpha(t))^o \vdash (\alpha(y))^o$ and $x = y \to (\alpha(x) \to \alpha(y))$. So,

$$\forall t(\alpha(t))^o, \alpha(x) \wedge \neg\alpha(y), x = y \vdash (\alpha(y))^o$$

$$\forall t(\alpha(t))^o, \alpha(x) \wedge \neg\alpha(y), x = y \vdash \neg\alpha(y)$$

$$\forall t(\alpha(t))^o, \alpha(x) \wedge \neg\alpha(y), x = y \vdash \alpha(y).$$

Consequently,

$$\forall t(\alpha(t))^o, \alpha(x) \wedge \neg\alpha(y) \vdash \neg(x = y)$$

$$\forall t(\alpha(t))^o \vdash \alpha(x) \wedge \neg\alpha(y) \to x \neq y. \ \blacksquare$$

Theorem 2.5.3 *In* $C_1^=$*:*

$$\forall y(\alpha(y) \leftrightarrow \exists x(x = y \wedge \alpha(x))$$

$$\forall y(\alpha(y) \leftrightarrow \forall x(x = y \to \alpha(x))$$

$$\forall y \exists x(x = y)$$

Theorem 2.5.4 *If we add to the postulates of* $C_1^=$ *the schema* $\neg(\alpha \wedge \neg\alpha)$*, we obtain* $C_0^=$*.*

Theorem 2.5.5 *Let* $\Gamma \vdash \alpha$*, and let* $\alpha_1, \ldots, \alpha_m$ *be the quantificationally prime components of the formula* α *and of the formulas in* Γ*. In this case,* $\Gamma \vdash \alpha$ *in* $C_0^=$ *iff* $\alpha_1^o, \ldots, \alpha_m^o, \Gamma \vdash \alpha$ *in* $C_1^=$*.*

Theorem 2.5.6 $C_\omega^=$ *is undecidable.*

Proof: $C_1^=$ contains $C_0^=$ just as $C_1^\star$ contains $C_0^\star$. So, $C_1^=$ is undecidable. $\blacksquare$

All valid schemas of $C_0^=$ that do not explicitly contain the symbol ¬ are valid in $C_1^=$.

Theorem 2.5.7 *If* $\Gamma \vdash \alpha$ *in* $C_0^=$*, and if in the formulas of* $\Gamma \cup \{\alpha\}$ *we replace* $\neg^\star$ *for* $\neg$*, thus obtaining the formula* $\alpha^\star$ *and the set of formulas* $\Gamma^\star$*, then* $\Gamma^\star \vdash \alpha^\star$ *in* $C_1^=$*.*

Proof: It suffices to note that $\neg^\star$ has the properties of classical negation. $\blacksquare$

Definition 2.5.1 $\exists ! x\alpha(x) =_{\text{def}} \exists y \forall x(y = x \leftrightarrow \alpha(x))$

Theorem 2.5.8 *In* $C_1^=$, $\vdash \forall x \exists ! y(x = y)$.

Theorem 2.5.9 *If* α *does not contain the equality symbol and* $\vdash \alpha$ *in* $C_1^=$, *then* $\vdash \alpha$ *in* $C_1^\star$.

Proof: Let D be a formal proof of α in $C_1^=$. There are only a finite number of applications of postulate (II') in D, say

$$x_1 = y_1 \to (\alpha_1(x_1) \leftrightarrow \alpha_1(y_1)), \ldots, x_k = y_k \to (\alpha_k(x_k) \leftrightarrow \alpha_k(y_k)).$$

Let u and v be variables that don't appear in the formulas of D. We denote by $K(u, v)$ the universal closure of the following formula, with respect to all free variables distinct from u and v:

$$(\alpha_1(x_1) \leftrightarrow \alpha_1(y_1)) \wedge \ldots \wedge (\alpha_k(x_k) \leftrightarrow \alpha_k(y_k)).$$

If we then replace subformulas of the form $x = y$ of formulas occurring in D by $K(x, y)$, we get a formal proof of α in $C_1^\star$, after a few suitable adaptations. ▮

As a consequence of the above theorem, we have that if $\Gamma \cup \{\alpha\}$ is a set of formulas of $C_1^\star$ such that $\Gamma \vdash \alpha$ in $C_1^=$, then $\Gamma \vdash \alpha$ also in $C_1^\star$.

The theorem 2.5.9 is important, for it shows that the quantificational schemes that are not deducible in $C_1^\star$ remain not deducible in $C_1^=$. So, the formulas $\alpha \wedge \neg\alpha \to \beta$, $\neg(\alpha(x) \wedge \neg\alpha(x))$ and $\exists x\alpha(x) \leftrightarrow \neg\forall x\neg\alpha(x)$ are not theorems of $C_1^=$, since they are not theorems of $C_1^\star$.

With regard to the calculi of the hierarchy (5), we can prove several interesting results. In what follows, we sum up some of these results, without proof.

Theorem 2.5.10 *The propositions* 2.5.1, 2.5.3, 2.5.4, 2.5.6 *and* 2.5.9 *hold for* $C_n^=$, $0 \leq n \leq \omega$.

Theorem 2.5.11 *If the quantificationally prime components of the formulas of* $\Gamma \cup \{\alpha\}$ *are* $\alpha_1, \ldots, \alpha_m$, *then if* $\Gamma \vdash \alpha$ *in* $C_0^=$, *we have that* $\alpha_1^o, \ldots, \alpha_m^o, \Gamma \vdash \alpha$ *in* $C_n^=$, $1 \leq n \leq \omega$.

Theorem 2.5.12 *Every calculus of the hierarchy (5) is strictly stronger than those following it.*

Theorem 2.5.13 *The calculi of the hierarchy* $C_n^=$, $1 \leq n < \omega$, *is finitely trivializable, while* $C_\omega^=$ *is not.*

2.6 Descriptions

In the calculi $C_n^=$, $1 \leq n \leq \omega$, we can introduce the description operator, ι, by means of a contextual definition, similarly to Russell's well-known definition in the case of $C_0^=$. Alternatively, the symbol ι can be introduced in the language of the calculi as a primitive symbol, satisfying the axioms presented below. This gives rise to a new hierarchy of calculi of descriptions:

$$\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n, \ldots, \mathcal{D}_\omega. \qquad (6)$$

The postulates to be added to those of $C_n^=$, $1 \leq n \leq \omega$, are the following ones (satisfying the usual restrictions; see [220]):

(D.1) $\forall x F(x) \to F(\iota y Q(y))$

(D.2) $\forall x(P(x) \leftrightarrow Q(x)) \to \iota x P(x) = \iota x Q(x)$

(D.3) $\iota x F(x) = \iota y F(y)$

(D.4) $P(\iota y Q(y)) \to \exists x P(x)$

(D.5) $\exists! x P(x) \to (\forall x((\iota x P(x) = x) \leftrightarrow P(x))$.

It can be proven that $\mathcal{D}_n$ is a conservative extension of $C_n^=$ $(0 \leq n \leq \omega)$. As above, we will focus the discussion on the first calculus of the hierarchy, namely $\mathcal{D}_1$. In particular, the following results are significant (and their proofs are adaptations of the classical case):

Theorem 2.6.1 *In* $\mathcal{D}_1$, $\vdash \iota x F(x) = \iota x F(x)$.

Theorem 2.6.2 *If* $\iota x F(x)$ *is free for x in* $F(x)$, *then* $\vdash \exists! x F(x) \to F(\iota x F(x))$.

Proof: From (D.5), we have $\vdash \exists! x F(x) \to \forall x((\iota x F(x) = x) \leftrightarrow F(x))$. Taking $\exists! x F(x)$ as hypothesis, we get $\forall x((\iota x F(x) = x) \leftrightarrow F(x))$. Then, from (D.1), $(\iota x F(x) = \iota x F(x)) \leftrightarrow F(\iota x F(x))$, hence $F(\iota x F(x))$. So, $\vdash \exists! x F(x) \to F(\iota x F(x))$. ▮

Theorem 2.6.3 *In* $\mathcal{D}_1$, $\vdash \forall y(\iota x(x = y) = y)$.

Theorem 2.6.4 *In* $\mathcal{D}_1$, *we have:*

1. $\vdash \forall y(\iota x P(x) = y \leftrightarrow y = \iota x P(x))$
2. $\vdash \forall y \forall x(\iota x P(x) = y \wedge y = z \to \iota x P(x) = z)$
3. $\vdash \forall x \forall z(x = \iota y Q(y) \wedge \iota y Q(y) = z \to x = z)$
4. $\vdash \forall z(\iota x P(x) = \iota y Q(y) \wedge \iota y Q(y) = z \to \iota x P(x) = z)$
5. $\vdash \forall y(\iota x P(x) = y \wedge y = \iota z R(z) \to \iota x P(x) = \iota z R(z))$
6. $\vdash \iota x P(x) = \iota y Q(y) \leftrightarrow \iota y Q(y) = \iota x P(x)$
7. $\vdash \iota x P(x) = \iota y Q(y) \wedge \iota y Q(y) = \iota z R(z) \to \iota x P(x) = \iota z R(z)$
8. $\vdash F(\iota x P(x)) \wedge \exists x(x = \iota x P(x) \to F(x))$
9. $\vdash F(\iota x P(x)) \leftrightarrow \forall x(x = \iota x P(x) \to F(x))$
10. $\vdash \iota x P(x) = \iota y Q(y) \wedge \iota z Q(z) = \iota t K(t) \to \iota x P(x) = \iota t K(t)$

Theorem 2.6.5 *Let $F(x)$ be a formula, and u and t be two terms (variables or descriptions) free for x in $F(x)$. Then,*

$$\vdash u = v \rightarrow (F(u) \rightarrow F(v)).$$

Theorem 2.6.6 *If y does not occur free in $P(x)$ and $\iota x P(x)$ is free for x in $P(x)$, then*

$$\vdash \exists! x P(x) \rightarrow (F(\iota x P(x)) \leftrightarrow \exists y (F(y) \land \forall x (x = y \leftrightarrow P(x)))).$$

Proof: Let us assume that $\exists! x P(x)$. So,

$$\forall x (\iota x P(x) = x \leftrightarrow P(x)).$$

Then, taking $F(\iota x P(x))$ as a hypothesis, we get

$$F(\iota x P(x)) \text{ and } \forall x (\iota x P(x) = x \leftrightarrow P(x)).$$

Hence,

$$F(\iota x P(x)) \land \forall x (x = \iota x P(x) \leftrightarrow P(x)),$$

and so,

$$\exists y (F(y) \land \forall x (x = y \leftrightarrow P(x)).$$

Consequently,

$$F(\iota x P(x)) \rightarrow \exists y (F(y) \land \forall x (x = y \leftrightarrow P(x))).$$

But, assuming that

$$\exists y (F(y) \land \forall x (x = y \leftrightarrow P(x)),$$

it results that

$$F(y) \land \forall x (x = y \leftrightarrow P(x)),$$

so

$$F(y) \text{ and } \forall x (x = y \leftrightarrow P(x)),$$

and hence we conclude that $y = \iota x P(x)$ and that $F(\iota x P(x))$. Then,

$$\exists y (F(y) \land \forall x (x = y \leftrightarrow P(x)) \rightarrow F(\iota x P(x)).$$

So,

$$\exists! x P(x) \rightarrow (F(\iota x P(x)) \leftrightarrow \exists y (F(y) \land \forall x (x = y \leftrightarrow P(x)))). \blacksquare$$

Theorem 2.6.7 *Let $A_1, \ldots, A_n$ be the quantificationally prime components of the formula F and of the formulas in Γ. In this case, if $\Gamma \vdash F$ in $\mathcal{D}_0$, then $A_1^o, \ldots, A_n^o, \Gamma \vdash F$ in $\mathcal{D}_1$, and conversely.*

Let us consider the set of formulas Γ of $\mathcal{D}_1$, and suppose that the finitely many descriptions that appear in these formulas are:

$\iota x_1 Q_1(x_1), \iota x_2 Q_1(x_2), \ldots$

$\iota x_1 Q_2(x_1), \iota x_2 Q_2(x_2), \ldots$

$\vdots$

which we admit ordered by some order relation. We can then associate to each of these descriptions one of the variables $t_0, \dots, t_{n-1}$, which do not appear in the formulas of Γ, in such a way that the association is a bijection. The ι-transforms of the formulas of Γ are the formulas obtained from the members of Γ by replacing in them the descriptions by their associated variables, and identifying the variables corresponding to descriptions, such as in the case of D.4.

Theorem 2.6.8 *If* $\vdash F$ *in* $\mathcal{D}_1$ *and* F *does not contain the symbols* $=$ *and* ι, *then* $\vdash F$ *in* $C_1^{\star}$.

Proof: If F can be proven in $\mathcal{D}_1$, it is deducible in $C_1^{=\star}$ from a finite number of formulas of types D.1–D.5 (see page 27 above), where $C_1^{=\star}$ is the calculus obtained from $C_1^{=}$ by introducing formation rules enabling descriptions as new terms in addition to variables. Let us denote by Δ a deduction of F in $C_1^{=\star}$ from formulas of types D.1–D.5. If we substitute, in such a deduction, formulas by their ι-transforms, we see that we get a valid deduction $\Delta^{\star}$ in $C_1^{=}$. But after such substitutions, the formulas of types D.1, D.3 and D.4 turn out to be axioms of $C_1^{=}$ and $\Delta^{\star}$ is a deduction of F from formulas of the forms (i) $\forall x(P(x) \leftrightarrow Q(x)) \rightarrow t_i = t_j$ and (ii) $\exists! x P(x) \rightarrow \forall x(t_k = x \leftrightarrow P(x))$. If in $\Delta^{\star}$ we identify pairs of variables, such as t_i and t_j, that appear free in the formulas of the type (i) above, then $\Delta^{\star}$ becomes a deduction $\Delta^{\star\star}$, which is still valid in $C_1^{=}$. But since the formulas of the specified type are axioms of $C_1^{=}$, they can be suppressed from the hypothesis of $\Delta^{\star\star}$. This yields a deduction $\Delta^{\star\star\star}$ of F in $C_1^{=}$ from formulas of the form (ii). However, to say that F is deducible in $C_1^{=}$ from formulas of the above type is to say that F is deducible in $C_1^{\star}$ from formulas of the following forms: $\forall x(x = x)$, $x = y \rightarrow (A(x) \rightarrow A(y))$ and $P(t_k)$. Therefore, taking into account theorem 2.5.9, we easily complete the proof. ▮

The calculi $\mathcal{D}_0, \mathcal{D}_2, \dots, \mathcal{D}_n, \dots, \mathcal{D}_\omega$ have properties similar to those of $\mathcal{D}_1$.

Theorem 2.6.9 *The Theorems 2.6.1 to 2.6.6 hold for every calculi* $\mathcal{D}_n$, $0 \leq n \leq \omega$.

Theorem 2.6.10 *If* $\vdash F$ *in* $\mathcal{D}_0$ *and* F *does not contain the symbol* ι, *then* $\vdash F$ *in* $C_0^{=}$.

Theorem 2.6.11 *The calculi* $\mathcal{D}_n$, $0 \leq n \leq \omega$ *are simply consistent.*[7]

Proof: By the previous theorem, $\mathcal{D}_0$ is simply consistent if $C_0^{=}$ is. Since the remaining calculi are sub-systems of $\mathcal{D}_0$, the theorem follows. ▮

Theorem 2.6.12 *Let* $0 \leq n < \omega$, *and let* $A_1, \dots, A_n$ *be the quantificationally prime components of the formula* F *and of the formulas in* Γ. *Then,* $\Gamma \vdash F$ *in* $\mathcal{D}_0$ *iff* $A_1^{(n)}, \dots, A_n^{(n)} \Gamma \vdash F$ *in* $\mathcal{D}_0$.

Theorem 2.6.13 *The calculi* $\mathcal{D}_n$, $0 \leq n < \omega$, *are undecidable.*

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[7] This means that for no formula A, both A and $\neg A$ are provable in these calculi ([159, p. 124]).

Theorem 2.6.14 *Every calculus of the hierarchy* $\mathcal{D}_n$, $0 \leq n \leq \omega$, *is strictly stronger than those that follow it.*

Theorem 2.6.15 $\mathcal{D}_n$, $0 \leq n < \omega$, *is finitely trivializable.*

Theorem 2.6.16 $\mathcal{D}_\omega$ *is not finitely trivializable.*

Theorem 2.6.17 *If* $\Gamma \vdash \alpha$ *in* $\mathcal{D}_n$ *and the formulas in* $\Gamma \cup \{\alpha\}$ *do not contain the description symbol, then* ‘$\Gamma \vdash \alpha$ *in* $C_n^=$, $0 \leq n \leq \omega$.

2.7 Semantics

We begin by sketching a two-valued semantics for C_1.[8] Here, $\mathcal{F}$ stands for the set of formulas of C_1; Γ and Δ designate subsets of $\mathcal{F}$, while $\overline{\Gamma}$ denotes the set of all formulas α such that $\Gamma \vdash \alpha$.

Recall that we say that a set Γ of formulas is trivial iff $\overline{\Gamma} = \mathcal{F}$; otherwise, it is non-trivial. Γ is inconsistent iff there is at least one formula α such that both α and $\neg\alpha$ belong to $\overline{\Gamma}$; otherwise, Γ is consistent. Finally, Γ is maximal non-trivial iff it is non-trivial and, for any formula α, if $\alpha \notin \Gamma$, then $\Gamma \cup \{\alpha\}$ is trivial.

Theorem 2.7.1 *If* Γ *is maximal non-trivial, then:*

1. $\Gamma \vdash \alpha \Leftrightarrow \alpha \in \Gamma$
2. $\alpha \in \Gamma \Rightarrow \neg^\star\alpha \notin \Gamma$
3. $\neg^\star\alpha \in \Gamma \Rightarrow \alpha \notin \Gamma$
4. $\alpha \in \Gamma$ or $\neg^\star\alpha \in \Gamma$
5. $\vdash \alpha \Rightarrow \alpha \in \Gamma$
6. $\alpha, \alpha^o \in \Gamma \Rightarrow \neg\alpha \notin \Gamma$
7. $\neg\alpha, \alpha^o \in \Gamma \Rightarrow \alpha \notin \Gamma$
8. $\alpha, \alpha \rightarrow \beta \in \Gamma \Rightarrow \beta \in \Gamma$
9. $\alpha^o \in \Gamma \Rightarrow \alpha \notin \Gamma$ or $\neg\alpha \notin \Gamma$
10. $\alpha^o \in \Gamma \Rightarrow (\neg\alpha)^o \in \Gamma$

Proof: We will prove only one side of the first property. Suppose that $\Gamma \vdash \alpha$ but $\alpha \notin \Gamma$. Then, since Γ is maximal non-trivial, $\Gamma \cup \{\alpha\} \vdash \alpha \wedge \neg^\star\alpha$. Hence $\Gamma \vdash \alpha \rightarrow (\alpha \wedge \neg^\star\alpha)$, and $\Gamma \vdash \neg^\star\alpha$. But, since $\Gamma \vdash \alpha$, then $\Gamma \vdash \alpha \wedge \neg^\star\alpha$, therefore Γ is trivial, which is absurd. ∎

Definition 2.7.1 *A valuation of* C_1 *is a mapping* $v : \mathcal{F} \mapsto \{1, 0\}$ *such that:*

1) $v(\alpha) = 0 \Rightarrow v(\neg\alpha) = 1$
2) $v(\neg\neg\alpha) = 1 \Rightarrow v(\alpha) = 1$
3) $v(\beta^o) = v(\alpha \rightarrow \beta) = v(\alpha \rightarrow \neg\beta) = 1 \Rightarrow v(\alpha) = 0$
4) $v(\alpha \rightarrow \beta) = 1 \Leftrightarrow v(\alpha) = 0$ *or* $v(\beta) = 1$
5) $v(\alpha \wedge \beta) = 1 \Leftrightarrow v(\alpha) = v(\beta) = 1$
6), $v(\alpha \vee \beta) = 1 \leftrightarrow v(\alpha) = 1$ *or* $v(\beta) = 1$
7) $v(\alpha^o) = v(\beta^o) = 1 \Rightarrow v((\alpha \vee \beta)^o) = v(\alpha \wedge \beta)^o) = v((\alpha \rightarrow \beta)^o) = 1$.

Theorem 2.7.2 *If* v *is a valuation of* C_1, *it has the following properties:* $v(\alpha) = 1 \Leftrightarrow v(\neg^\star\alpha) = 0$, $v(\alpha) = 0 \Leftrightarrow v(\neg^\star\alpha) = 1$, $v(\alpha) = 0 \Leftrightarrow v(\alpha) = 0$ *and* $v(\neg\alpha) = 1$, $v(\alpha) = 1 \Leftrightarrow v(\alpha) = 1$ *or* $v(\neg\alpha) = 0$.

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[8] The semantics of valuations, conceived as a general semantic method, was developed by the first author of this paper in the 1960s, in his Logic Seminar at Federal University of Paraná, Brazil.

Definition 2.7.2 *(i) A valuation v is singular if there exists at least one formula α such that $v(\alpha) = v(\neg\alpha) = 1$; otherwise, v is normal. (ii) A formula α is valid if for every valuation v, $v(\alpha) = 1$. (iii) A valuation v is a model for a set Γ of formulas if $v(\alpha) = 1$ for any $\alpha \in \Gamma$. (iv) A formula α is a semantic consequence of Γ if every model v of Γ is such that $v(\alpha) = 1$; in this case, we write $\Gamma \models \alpha$. In particular, $\models \alpha$ – which abbreviates $\emptyset \models \alpha$ – means that α is valid.*

Theorem 2.7.3 (Soundness) $\Gamma \vdash \alpha \Rightarrow \Gamma \models \alpha$ *(in particular,* $\vdash \alpha \Rightarrow \models \alpha$*).*

Proof: As in classical logic. ▮

Lemma 2.7.1 *Every non-trivial set of formulas is contained in a maximal non-trivial set.*

Proof: By adapting the corresponding proof in classical logic. ▮

Corolary 2.7.1 *There exist maximal non-trivial inconsistent sets.*

Proof: It is easy to see that $\{\alpha, \neg\alpha\}$ is inconsistent but non-trivial. By the preceding Lemma, it is contained in a maximal non-trivial set of formulas, which is inconsistent. ▮

Lemma 2.7.2 *Every maximal non-trivial set of formulas* Γ *has a model.*

Proof: Define a mapping $v : \mathcal{F} \mapsto \{0, 1\}$ as follows: for every formula α, if $\alpha \in \Gamma$, then $v(\alpha) = 1$, and $v(\alpha) = 0$ otherwise. It is then easy to see that v satisfies all the conditions in the definition of a valuation. ▮

Corolary 2.7.2 *Any non-trivial set of formulas has a model.*

Corolary 2.7.3 *There are singular valuations; that is, valuations that assign value 1 to contradictory formulas. (A valuation that is not singular is called normal.)*

Proof: The set $\{\alpha, \neg\alpha\}$ is inconsistent but non-trivial, hence it is contained in a maximal non-trivial set, as shown above. But this set has a model, which is a singular valuation. ▮

Theorem 2.7.4 (Completeness) $\Gamma \models \alpha \Rightarrow \Gamma \vdash \alpha$ *(in particular,* $\models \alpha \Rightarrow \vdash \alpha$*).*

Proof: Let us suppose that $\Gamma \nvdash \alpha$. In this case, $\Gamma \cup \{\neg^\star\alpha\}$ is non-trivial and has a model v. So, there is a model v of Γ such that $v(\alpha) = 0$, which is absurd.▮

Theorem 2.7.5 *There are inconsistent (but non-trivial) sets of formulas that have models.*

Definition 2.7.3 *Let* $\Delta =_{\mathrm{def}} \{\alpha^o \in \mathcal{F} : \vdash \alpha\}$*. Then* Γ *is said to be strongly non-trivial if* $\Gamma \cup \Delta$ *is non-trivial.* Γ *is said to be strictly non-trivial if* $\Gamma \cup \Delta$ *is non-trivial.*

Theorem 2.7.6 *There exist sets of formulas that are non-trivial and sets of formulas that are strictly non-trivial.*

Proof: To prove the first part of the theorem, suppose that Δ is the set $\{\alpha^o \in \mathcal{F} : \vdash \alpha\}$. Then Δ is consistent, which implies that it is also non-trivial. So, Δ is contained in a maximal non-trivial set Δ'. Let Δ'' be $\Delta' - \Delta$. Then Δ'' is strongly non-trivial. ▮

We will now discuss a byproduct of the semantics for C_1 sketched above, namely, the decidability of this calculus.

2.7.1 The decidability of C_1.

Definition 2.7.4 (Quasi-matrix) *We call quasi-matrices the tables constructed according to the instructions given below, applicable to each formula of* C_1*(see [9]). Given a formula* α*:*

1. Make a list of all propositional variables that appear in α, and arrange them in a line.

2. Under the propositional variables, place all possible distributions of 0s and 1s that can be attributed to the variables, as usual.

3. Make a list of the negations of propositional variables appearing in the formula. And calculate their truth-value, in each line, as follows: (i) if a variable has value 0, its negation will have truth-value 1; (ii) if the variable has value 1, bifurcate the line in which it appears, by writing in the first part the value 0 for the negation and, in the second part, the value 1 for the negation. Before a bifurcation, the truth-values must be the same for the two resulting lines.

4. Complete the previous list, by adding the sub-formulas of α and the negations of proper sub-formulas. And calculate, for each line, the truth-value of each sub-formula of α. If it is a proper sub-formula, calculate the value of its negation (whose proper sub-formulas and their negations have already been listed and calculated), as follows: (i) When no negations are involved, proceed as in the truth-tables for the classical propositional calculus; (ii) If any of the formulas in consideration is a negation – and so of the form $\neg\alpha'$ – write the truth-value 1 under it in those lines where α' has value 0. And in the lines where α' has value 1, proceed as follows: (i') If α' is of the form $\neg\beta$, then check if the value of β is the same as the value of $\neg\beta$; in this case, bifurcate the line by writing the value 0 in the first part and the value 1 in the second. If the value of β is distinct from the value of $\neg\beta$, simply write 0. (ii') If α' is of the form $\beta \bowtie \gamma$, where $\bowtie \in \{\rightarrow, \vee, \wedge\}$, there are two cases to consider: (a) α' is of the form $\delta \wedge \neg\delta$ or of the form $\neg\delta \wedge \delta$. In this case, write the value 0 for the formula α'; (b) α' is neither of the form $\delta \wedge \neg\delta$ nor of the form $\neg\delta \wedge \delta$. In this case, check if the value of β is equal to the value of $\neg\beta$ or if the value of γ is equal to the value of $\neg\gamma$. In the positive case, bifurcate the line, by writing the value 0 in the first part and, in the second, write 1. If the value of β is distinct from the value of $\neg\beta$, simply write 0.

We will exemplify this definition below. Before that, we state the following lemma:

Lemma 2.7.3 $v : \mathcal{F} \mapsto \{0, 1\}$ *is a valuation iff:*

1. $v(\neg\alpha) = 0 \Rightarrow v(\alpha) = 1$

2. $v(\neg\neg\alpha) = 1 \Rightarrow v(\alpha) = 1$

3. $v(\beta^o) = v(\alpha \to \beta) = v(\alpha \to \neg\beta) = 1 \Rightarrow v(\alpha) = 0$

4. $v(\alpha \to \beta) = 1 \Leftrightarrow v(\alpha) = 0$ or $v(\beta) = 1$

5. $v(\alpha \wedge \beta) = 1 \Leftrightarrow v(\alpha) = v(\beta) = 1$

6. $v(\alpha \vee \beta) = 1 \Leftrightarrow v(\alpha) = 1$ or $v(\beta) = 1$

7. $v((\alpha \wedge \beta)^o) = 0 \Leftrightarrow v(\alpha^o) = 0$ or $v(\beta^o) = 0$

8. $v((\alpha \vee \beta)^o) = 0 \Leftrightarrow v(\alpha^o) = 0$ and $v(\beta^o) = 0$

9. $v((\alpha \to \beta)^o) = 0 \Leftrightarrow v(\alpha^o) = 0$ or $v(\beta^o) = 1$

Lemma 2.7.4 $v(\alpha^o) = 0 \Leftrightarrow v(\alpha) = v(\neg\alpha) = 1$

Proof: (a) $v(\alpha^o) = 0$ implies that $v(\alpha \wedge \neg\alpha) = 1$ and $v(\alpha) = v(\neg\alpha) = 1$. (b) Suppose that $v(\alpha) = v(\neg\alpha) = 1$. If $v(\alpha^o) = 1$, then $v(\alpha) = v(\neg\alpha) = v(\alpha^o) = 1$, that is, $v(\alpha) = v(\neg^\star\alpha) = 1$, and v would not be a valuation. Hence $v(\alpha^o) = 0$ and, therefore, $v(\alpha) = v(\neg\alpha) = 1$. ▮

Lemma 2.7.5 $v : \mathcal{F} \mapsto \{0, 1\}$ *is a valuation iff the conditions 1-6 of Lemma 2.7.3 hold and:*

7i. $v((\alpha \to \beta)^o) = 0 \Leftrightarrow v(\alpha) = v(\neg\alpha) = 1$ or $v(\beta) = v(\neg\beta) = 1$

7ii. $v((\alpha \wedge \beta)^o) = 0 \Leftrightarrow v(\alpha) = v(\neg\alpha) = 1$ or $v(\beta) = v(\neg\beta) = 1$

7ii. $v((\alpha \vee \beta)^o) = 0 \Leftrightarrow v(\alpha) = v(\neg\alpha) = 1$ or $v(\beta) = v(\neg\beta) = 1$

Definition 2.7.5 *Let v be a valuation and α a formula. Then v_α is called the restriction of v to the set of sub-formulas of α and the negations of proper sub-formulas of α.*

Lemma 2.7.6 *For every valuation v and formula α, $v(\alpha) = v_\alpha(\alpha)$.*

Definition 2.7.6 *Let v be a valuation and Γ a set of formulas. Then v_Γ is the restriction of v to the set Γ.*

Definition 2.7.7 *We say that a line of a quasi-matrix corresponds to v_Γ if $v_\Gamma(\alpha)$ is the value corresponding to α in that line for every $\alpha \in \Gamma$, where Γ is the set of all formulas of the matrix.*

Lemma 2.7.7 *Given a quasi-matrix Q, then for every valuation v there exists a line of Q that corresponds to v_Γ, where Γ is the set of all formulas of Q.*

Proof: By induction on the number of columns of Q. ▮

Definition 2.7.8 *Let Q be a quasi-matrix for a formula α, and let Γ be the set of all sub-formulas and negations of proper sub-formulas of α. Let k be a line of Q and $k(\alpha)$ be the value attributed to α in k. We call $\Delta(\Gamma, k)$ the set of formulas such that, for every formula α:*

(I) *If* $\alpha \in \Gamma$*, then* $\alpha \in \Delta(\Gamma, k)$ *iff* $k(\alpha) = 0$.

(II) *If* $\alpha \notin \Gamma$*, then* $\alpha \in \Delta(\Gamma, k)$ *iff*

- *a)* α *is atomic, or*
- *b)* α *is of the form* $\neg\alpha_1$ *and* $\alpha_1 \notin \Delta(\Gamma, k)$*, or*
- *c)* α *is* $\alpha_1 \wedge \alpha_2$ *and* $\alpha_1 \in \Delta(\Gamma, k)$ *and* $\alpha_2 \in \Delta(\Gamma, k)$*, or*
- *d)* α *is* $\alpha_1 \vee \alpha_2$ *and* $\alpha_1 \in \Delta(\Gamma, k)$ *and* $\alpha_2 \in \Delta(\Gamma, k)$*, or*
- *e)* α *is* $\alpha_1 \rightarrow \alpha_2$ *and* $\alpha_1 \notin \Delta(\Gamma, k)$ *and* $\alpha_2 \in \Delta(\Gamma, k)$.

Some properties of the set $\Delta(\Gamma, k)$ are the following:

1. $\neg\alpha \in \Delta(\Gamma, k) \Rightarrow \alpha \notin \Delta(\Gamma, k)$

2. $\alpha \in \Delta(\Gamma, k) \Rightarrow \neg\neg\alpha \in \Delta(\Gamma, k)$

3. $\neg^{\star}\alpha \in \Delta(\Gamma, k) \Leftrightarrow \alpha \notin \Delta(\Gamma, k)$

4. $\alpha \rightarrow \beta \notin \Delta(\Gamma, k) \Leftrightarrow \alpha \in \Delta(\Gamma, k)\,\text{or}\,\beta \notin \Delta(\Gamma, k)$

5. $\alpha \in \Delta(\Gamma, k)\,\text{or}\,\beta \in \Delta(\Gamma, k) \Leftrightarrow \alpha \wedge \beta \in \Delta(\Gamma, k)$

6. $\alpha \notin \Delta(\Gamma, k)\,\text{or}\,\beta \notin \Delta(\Gamma, k) \Leftrightarrow \alpha \vee \beta \notin \Delta(\Gamma, k)$

7. $(\alpha \bowtie \beta)^o \in \Delta(\Gamma, k) \Rightarrow \alpha \notin \Delta(\Gamma, k)\,\text{and}\,\neg\alpha \notin \Delta(\Gamma, k),\ \text{or}\,\beta \notin \Delta(\Gamma, k)\,\text{and}\,\neg\beta \notin \Delta(\Gamma, k)$, where $\bowtie \in \{\rightarrow, \wedge, \vee\}$.

Lemma 2.7.8 (A. Loparić) *For every line k of a quasi-matrix Q, there is a valuation* v *such that* v_Γ *corresponds to k, where* Γ *is the set of formulas of Q.*

Proof: Let v be a function from $\mathcal{F}$ in $\{0, 1\}$ such that, for every $\alpha \in \mathcal{F}$, $v(\alpha) = 0$ if $\alpha \in \Delta(\Gamma, k)$, and $v(\alpha) = 1$ if $\alpha \notin \Delta(\Gamma, k)$. Then, by the properties 1-7 above of the set $\Delta(\Gamma, k)$, v is a valuation. Since v_Γ and k are the same, we can say that there exists a valuation v such that v_Γ corresponds to k. ▮

Theorem 2.7.7 (M. Fidel) *The calculus* C_1 *is decidable.*

Proof: Consequence of Lemmas 2.7.6, 2.7.7 and 2.7.8. The formula α is a theorem of C_1 iff in any quasi-matrix for α, the last column has only 1's. In effect, in this case, for every valuation v, we have that $v(\alpha) = v_\alpha(\alpha) = 1$. ▮

Let us exemplify the method presented here by showing that $\neg(\alpha \vee \beta) \rightarrow \neg\alpha \wedge \neg\beta$ is not valid in C_1:

α	β	$\neg\alpha$	$\neg\beta$	$\alpha \vee \beta$	$\neg(\alpha \vee \beta)$	$\neg\alpha \wedge \neg\beta$	$\neg(\alpha \vee \beta) \to \neg\alpha \wedge \neg\beta$
0	0	1	1	0	1	1	1
1	0	0	1	1	0	0	1
		1	1	1	0	1	1
					1	1	1
0	1	1	0	1	0	0	1
			1	1	0	1	1
					1	1	1
1	1	0	0	1	0	0	1
			1	1	0	0	1
					1	0	0
		1	0	1	0	0	1
					1	0	0
			1	1	0	1	1
					1	1	1

The extension of the semantics of C_1 to the systems $C_n, 2 \leq n < \omega$, is immediate. All definitions and theorems are the same, with obvious adaptations (for instance, $\neg^\star$ becomes $\neg^{(n)}$, and α^o becomes $\alpha^{(n)}$). When constructing quasi-matrices, if α is of the form $\beta^{(n-1)} \wedge \neg\beta^{(n-1)}$, or of the form $\neg\beta^{(n-1)} \wedge \beta^{(n-1)}$, we must write 0 for the formula α. We must have $v(\neg\alpha) = 0$, otherwise, due to clause 7 of Definition 2.7.1, $v(\beta \wedge \neg^{(n)}\beta) = 1$, and then v would not be a valuation. Note that this clause is precisely the one that characterizes the quasi-matrices of the systems $C_n, 1 \leq n < \omega$.

As an additional example, to see that the schema $(\alpha^{(n-1)} \wedge \neg\alpha^{(n-1)})^{(n)}$ is valid in C_n, but not in C_m, for $m > n$, it suffices to show that the schema $(\alpha \wedge \neg\alpha)^o$ is valid in C_1, but not in C_2. In fact, we have in C_1:

α	$\neg\alpha$	$\alpha \wedge \neg\alpha$	$\neg(\alpha \wedge \neg\alpha)$	$(\alpha \wedge \neg\alpha) \wedge \neg(\alpha \wedge \neg\alpha)$	$\neg((\alpha \wedge \neg\alpha) \wedge \neg(\alpha \wedge \neg\alpha))$
0	1	0	1	0	1
1	0	0	1	0	1
	1	1	0	0	1

However, in C_2, we have:

α	$\neg\alpha$	$\alpha \wedge \neg\alpha$	$\neg(\alpha \wedge \neg\alpha)$	$(\alpha \wedge \neg\alpha) \wedge \neg(\alpha \wedge \neg\alpha)$	$\neg((\alpha \wedge \neg\alpha) \wedge \neg(\alpha \wedge \neg\alpha))$
0	1	0	1	0	1
1	0	0	1	0	1
	1	1	0	0	1
			1	1	0
					1

It is also possible to show that C_ω is decidable; see [173] and section 6.

2.7.2 Semantics for $C_1^=$

The semantics sketched above can be extended to the quantificational calculi described earlier. Let us begin by defining an *interpretation* for $C_1^=$ (in fact, for the language of

$C_1^=$) as a pair $I = \langle D, \rho \rangle$, where D is a non-empty set and ρ is a mapping such that: (i) ρ associates an element $\rho(a) \in D$ to each individual constant a of the language of $C_1^=$; (ii) for each n-ary functional symbol f of the language of $C_1^=$, ρ associates an n-ary function from D^n to D, and (iii) to each n-ary predicate symbol P, other than identity, ρ associates an n-ary relation on D; (iv) to the identity symbol $=$, ρ associates the diagonal of D, namely, the set $\Delta_D =_{\text{def}} \{\langle x, x\rangle : x \in D\}$.[9]

If I is an interpretation for $C_1^=$, then for each element of D, we choose a new individual constant c, the name of the element (as usual, different names are chosen for different elements). This new language, as is well known, is called the *diagram language* of $C_1^=$ relatively to I ([228, p. 18]), and we refer to it as $\mathrm{D}C_1^=$.

If I is an interpretation for $C_1^=$, a *valuation* v of $C_1^=$ is a map from the set of sentences of $\mathrm{D}C_1^=$ in $\{0, 1\}$, defined as follows:

(1–7) Clauses 1-7 of Definition 2.7.1

8) $v(\forall x \alpha(x)) = 1$ iff for any individual constant c of $\mathrm{D}C_1^=$, $v(\alpha(c)) = 1$

9) $v(\exists x \alpha(x)) = 1$ iff there exists a constant c of $\mathrm{D}C_1^=$ so that $v(\alpha(c)) = 1$

10) $v(\forall x(\alpha(x))^o) = 1 \Rightarrow v((\forall x \alpha(x))^o) = v((\exists x \alpha(x))^o) = 1$

11) If α and β are congruent, then $v(\alpha) = v(\beta)$

12) $v(c = c') = 1$ iff $\rho(c) = \rho(c')$

13) $v(c = c') = 1$ and $v(\alpha(c)) = 1$, then $v(\alpha(c')) = 1$

We say that a valuation v *satisfies* a sentence α of $\mathrm{D}C_1^=$ (and, in particular, of $C_1^=$) if $v(\alpha) = 1$. v is a *model* for a set Γ of sentences if $v(\alpha) = 1$ for every $\alpha \in \Gamma$, and α is a *semantic consequence* of Γ (in symbols, $\Gamma \vDash \alpha$) if $v(\alpha) = 1$ for every model v of Γ.

Theorem 2.7.8 (Soundness) $\Gamma \vdash \alpha \Rightarrow \Gamma \vDash \alpha$, *where* $\Gamma \cup \{\alpha\}$ *is a set of sentences.*

Proof: As in the classical case, using induction on the length of the proof of α from Γ. ▮

We say that Γ is *trivial* if $\Gamma \vdash \alpha$, for every α; otherwise, Γ is *non-trivial*. Γ is *inconsistent* if there exists α such that $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$; otherwise, Γ is *consistent*. Finally, a set Γ os sentences is a *Henkin set* if for any formula $\alpha(x)$ with just one free variable, there exists a constant c of the language of $C_1^=$ such that $\Gamma \vdash \exists x \alpha(x) \to \alpha(c)$.

Theorem 2.7.9 *If* Γ *is non-trival (and a Henkin set), then it is contained in a maximal non-trivial (and also a Henkin set) set of sentences.*

Proof: As in the classical case. ▮

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[9] As we noted earlier, the language may contain neither individual constants nor functional symbols.

Theorem 2.7.10 *If* Γ *is a maximal non-trivial Henkin set of sentences, then:*

(1) $\alpha \to \beta \in \Gamma \Leftrightarrow \alpha \notin \Gamma$ *or* $\beta \in \Gamma$ (2) $\alpha \wedge \beta \in \Gamma \Leftrightarrow \alpha, \beta \in \Gamma$
(3) $\alpha \vee \beta \in \Gamma \Leftrightarrow \alpha \in \Gamma$ *or* $\beta \in \Gamma$ (4) $\Gamma \vdash \alpha \Leftrightarrow \alpha \in \Gamma$
(5) $\alpha \in \gamma \Leftrightarrow \neg^{\star}\alpha \notin \Gamma$ (6) $\neg\alpha, \alpha^{o} \in \Gamma \Rightarrow \alpha \notin \Gamma$
(7) $\alpha, \alpha^{o} \in \Gamma \Rightarrow \neg\alpha \notin \Gamma$ (8) $\alpha \in \Gamma$ *or* $\neg^{\star}\alpha \in \Gamma$
(9) $\alpha, \alpha \to \beta \in \Gamma \Rightarrow \beta \in \Gamma$ (10) $\alpha^{o} \in \Gamma \Rightarrow (\neg\alpha)^{o} \in \Gamma$
(11) $\alpha^{o}, \beta^{o} \in \Gamma \Rightarrow (\alpha \to \beta)^{o}, (\alpha \wedge \beta)^{o}, (\alpha \vee \beta)^{o} \in \Gamma$
(12) $\forall x\alpha(x) \in \Gamma$*iff for any constant c of* $C_1^{=}$,$\alpha(c) \in \Gamma$
(13) $\exists x\alpha(x) \in \Gamma$ *iff for some constant c of* $C_1^{=}$, $\alpha(c) \in \Gamma$
(14) $\forall x(\alpha(x))^{o} \in \Gamma \Rightarrow (\forall x\alpha(x))^{o} \in \Gamma$ *and* $(\exists x\alpha(x))^{o} \in \Gamma$
(15) *If* α *and* β *are congruent, then* $\alpha \leftrightarrow \beta \in \Gamma$
(16) $c = c', \alpha(c) \in \Gamma \Rightarrow \alpha(c') \in \Gamma$

Proof: As in the classical case, by using the strong negation $\neg^{\star}$ instead of the weak negation $\neg$. ▮

Theorem 2.7.11 *If* Γ *is a Henkin non-trivial set of sentences (either consistent or inconsistent), then* Γ *has a model.*

Proof: Consequence of the preceding theorem. ▮

Corolary 2.7.4 *Every non-trivial set of sentences of the language of* $C_1^{=}$ *has a model.*

Theorem 2.7.12 (Completeness) $\Gamma \models \alpha \Rightarrow \Gamma \vdash \alpha$

Theorem 2.7.13 (Löwenheim-Skolem) *If* Γ *has an infinite model, then it has an infinite denumerable model.*

Other results can also be obtained, so that it is possible to construct a paraconsistent model theory. For instance, E. A. Alves has developed such a theory for a variant of the calculus C_1 to which he introduced an additional axiom, namely, $\neg\neg\alpha \leftrightarrow \alpha$ (see [10]). (Recall that $\alpha \to \neg\neg\alpha$ is not a theorem of C_1; see Theorem 2.1.3.) The preceding results can also be adapted in order to be applied to the calculi C_n, $1 \leq n < \omega$, as well as to the calculi with descriptions $\mathcal{D}_n$, $1 \leq n < \omega$.

2.8 Syllogism and paraconsistency

Similarly to the case of traditional syllogistic, which was interpreted within classical monadic predicate calculus, it is possible to develop a paraconsistent syllogistic. It is based, for instance, on the monadic calculus corresponding to the paraconsistent predicate logic $C_1^{\star}$. In order to do that, it suffices to translate the propositions A, I, E and O into $C_1^{\star}$. The translations (based on the classical case) are as follows:

Aab	$\forall x(a(x) \to b(x))$
Iab	$\exists x(a(x) \wedge b(x))$
Eab	$\forall x(a(x) \to \neg b(x))$

Oab $\exists x(a(x) \wedge \neg b(x))$

Two remarks are in order here. (1) The valid positive deductions in $C_0^\star$, the classical predicate calculus, are also valid in $C_1^\star$; that is, when no explicit negation is involved, the positive deductions of $C_0^\star$ and C_1^* are the same. (2) In $C_1^\star$, one can find paraconsistent predicates, for which there are objects that satisfy these predicates and, at the same time, do not satisfy them. In other words, for some predicate p, the following holds:

$$\exists x(p(x) \wedge \neg p(x)).$$

Thus, based on arguments similar to the ones found in the classical case, it is possible to verify the validity of inferences, and one changes accordingly the theories of opposition, conversion, immediate inferences and syllogism. (Each predicate in the universe of discourse has three parts: of the elements that satisfy it, of those that do not satisfy it, and of those that simultaneously satisfy it and do not satisfy it. Simple graphics supply then evidence for the validity, or for the invalidity, of certain inferences and conversions.)

Based on this approach, one can prove the following result. In the paraconsistent logic $C_1^\star$, all modes of the first and of the third figures of the syllogism are valid; none of the second is valid; and of the fourth, just Bramantip and Dimaris modes are valid. It is worth mentioning that since $C_1^\star$ has a strong negation, of a classical trend, and if such negation is adopted in the interpretation of syllogistic reasoning, then the classical theory is obtained. As is known, Łukasiewicz has axiomatized the theory of categorical syllogism, based on the classical propositional calculus and admitting as specific axioms certain categorical propositions, as well as some appropriate definitions. Based on the paraconsistent propositional calculus, for instance the calculus C_1, it is also possible to formulate an axiomatics for paraconsistent syllogistic, articulated in parallel lines to the theory just outlined. Moreover, we should note that there are further extensions or modifications of the Aristotelian syllogistic that also admit paraconsistent versions, such as Hamilton's, De Morgan's and Gergone's.

2.8.1 Aristotle's syllogistic and paraconsistency

A surprising result can be sketched as follows. The theory of syllogism can be developed in an appropriate language so that the only formulas turn out to be expressions corresponding to the categorical propositions A, E, I and O. We can then interpret the valid syllogisms (given by the four figures of Aristotelian syllogistic) as providing inference rules, and adopt the Aristotelian definition of contradictory propositions to mean contradictory formulas of our language. Thus, it is easy to see that, based on such a language, we can define contradictory theories (that is, those which involve contradictory formulas in the above sense), which are non-trivial (in the sense that not all formulas are theorems). This means that, with a suitable interpretation, Aristotle's syllogistic can be seen as a paraconsistent logic, since from two contradictory premisses we cannot deduce any proposition whatsoever. Analogously, Aristotle's modal syllogistic constitutes also a paraconsistent modal logic.

3 Paraconsistent set theory

Cantor's naive set theory was based mainly on two fundamental principles: the postulate of extensionality (if the sets x and y have the same elements, then they are equal), and the postulate of comprehension or separation (every property determines a set, composed of the objects that have this property). The latter postulate, in the standard (first-order) language of set theory, becomes the following formula (or schema of formulas):

$$\exists y \forall x (x \in y \leftrightarrow F(x)). \tag{7}$$

Now, it is enough to replaces the formula $F(x)$ in (7) by $x \notin x$ to derive Russell's paradox. That is, the principle of comprehension (7) entails an inconsistency. Thus, if one adds (7) to first-order logic, conceived as the logic of a set-theoretic language, a trivial theory is obtained. There are also other paradoxes, such as those of Curry and Moh Schaw-Kwei, that indicate that (7) is trivial or, more precisely, trivializes set theory if its underlying logic is classical, even ignoring negation.[10] In other words, classical positive logic is incompatible with (7); the same holds also for several other logics, such as the intuitionistic one.

Classical set theories are distinguished by the restrictions they impose on (7), so that the paradoxes are avoided. In order that the theory thus obtained does not become too weak, some further axioms, besides extensionality and comprehension (with appropriate restrictions), are added, depending on the particular case. Thus, for instance, in Zermelo-Fraenkel (ZF), comprehension is formulated in the following way:

$$\forall z \exists y \forall x (x \in z \leftrightarrow (x \in \ \wedge F(x))), \tag{8}$$

where the variables are subject to the usual conditions. In ZF, then, $F(x)$ determines the subset of the elements of the set z that have the property F (or satisfy the formula $F(x)$). In the Kelly-Morse system, on the other hand, comprehension is formulated as follows:

$$\exists y \forall x (x \in y \leftrightarrow (F(x) \wedge \exists z (x \in z)). \tag{9}$$

In Quine's system NF, in turn, the notion of stratification is employed, and the schema of comprehension is written like (7), provided that $F(x)$ is stratified (and the standard conditions regarding the variables are met).

However, we can ask whether it is possible to examine the problem from a different viewpoint: what is needed to maintain the schema (7) *without* restrictions (disregarding the conditions on the variables)? The answer is immediate: one should change the underlying logic, so that (7) does not inevitably lead to trivialization. The comprehension schema, without strong restrictions, leads to contradictions. Hence, such a logic has to

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[10] Let us exemplify this fact with Curry's paradox. In (7), substitute for $F(x)$ the expression $x \in x \rightarrow \alpha$, where α is an arbitrary formula in which y does not appear free. Then, by (7), we get $\exists y \forall x (x \in y \leftrightarrow (x \in x \rightarrow \alpha))$. In honor of Curry, let us call this set y, c. In this case, $\forall x (x \in c \leftrightarrow (x \in x \rightarrow \alpha))$. But then $c \in c \leftrightarrow (c \in c \rightarrow \alpha)$, hence (i) $c \in c \rightarrow (c \in c \rightarrow \alpha)$ and (ii) $(c \in c \rightarrow \alpha) \rightarrow c \in c)$. However, the law of contraction, $(\gamma \rightarrow (\gamma \rightarrow \beta)) \rightarrow (\gamma \rightarrow \beta)$, holds. So, (i) entails (iii): $c \in c \rightarrow \alpha$. From (ii) and (iii), we get $c \in c$. Finally, from this last sentence and from (iii), by *modus pones*, we obtain α. This shows that (7) entails triviality even without negation.

be paraconsistent. It was slowly verified that there are infinitely many ways of weakening the classical restrictions imposed on the comprehension schema, each of them corresponding to distinct categories of paraconsistent logic. Furthermore, extremely weak logics have been formulated as the underlying logics of set theories in which (7) is used without any restrictions (not related to the choice of variables).

An important point is that several paraconsistent set theories contain the classical one, in Zermelo-Fraenkel's, Kelly-Morse's or Quine's formulations. Hence, paraconsistency goes beyond the classical domain, and allows for, among other things, the reconstruction of traditional mathematics (see [110], [78], [111], [191]). It is fair to claim that paraconsistent theories extend the classical ones, just as Poncelet's imaginary geometry comprises standard Euclidean geometry.

Moreover, these considerations suggest a difficulty in the foundations of logic. Classical elementary logic (in fact, it is enough to consider only positive logic) and the comprehension postulate seem to be both evident. We may even claim that they are equally evident, or intuitive. However, they are mutually incompatible. This constitutes, thus, an intriguing case of incompatible evidences, which generates, in particular, a difficulty for classical logic.

Without presenting a detailed philosophical analysis, we just note that classical and paraconsistent theories develop different approaches to the handling of inconsistencies. Interestingly, the paraconsistent approach contributes to a better understanding of the classical position itself (by helping, e.g., to clarify the benefits and limitations the position has). The paraconsistent approach also provides a clearer understanding of negation (e.g., by distinguishing different meanings of this notion), and the approach articulates a more realistic perception of the possibility of discourse even if the principle of non-contradiction is partially left behind.

3.1 The systems $\mathcal{NF}_n$, $1 \leq n \leq \omega$

Here we begin by describing $\mathcal{NF}_1$.[11]The underlying logic of $\mathcal{NF}_1$ is the calculus $C_1^=$, that is, we assume the language of this calculus plus the propositional postulates $\rightarrow_1$ to $\neg_4$ (see page 9), the postulates (I)-(VII) for the predicate calculus (see page 20), and the postulates for equality (I') and (II') (see page 24). The specific postulates of $\mathcal{NF}_1$ are:

(NF1) (Extensionality) $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$

(NF2) (Comprehension) $\exists y \forall x (x \in y \leftrightarrow \alpha(x))$, where x and y are distinct variables, y does not occur free in $\alpha(x)$ and this formula is either stratified or is of the form $x \notin x$.[12]

If we add to $C_1^=$ the schema $\neg(\alpha \wedge \neg\alpha)$, we get the classical first-order predicate calculus with identity. The system NF of Quine is obtained by adding to this calculus the postulates (NF1) and (NF2) above, provided that the latter postulate meets the condition that $\alpha(x)$ be stratified. We denote Quine's system by $\mathcal{NF}_0$.

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[11] Caiero and de Souza have developed a paraconsistent version of the *ML* system in [55].

[12] We recall that a formula is stratified if it is possible to replace each variable occurring in it by a numeral in the following manner: we replace everywhere the same variable by the same numeral so that, for each occurrence of $\in$, the numeral immediately following $\in$ is the immediate successor of the numeral immediately preceding $\in$ [144, p. 213].

In order to introduce the set theories $\mathcal{NF}_n$, $1 \leq n < \omega$, we employ the calculus $C_n^=$, $1 \leq n < \omega$, as their underlying logics plus the specific postulates above. Several results can now be established:

Theorem 3.1.1 *$\mathcal{NF}_n$, $1 \leq n < \omega$, contains $\mathcal{NF}_0$.*

Proof: It follows from the fact that if α is a theorem of $\mathcal{NF}_0$, and if we replace all occurrences of $\neg$ in this formula by $\neg^{(n)}$, obtaining α', then α' is provable in $\mathcal{NF}_n$, $1 \leq n < \omega$, where $\neg^{(n)}\alpha$ is the formula $\neg\alpha \wedge \alpha^{(n)}$.▮

Theorem 3.1.2 *If $\mathcal{NF}_n$, $1 \leq n < \omega$, is non-trivial, then $\mathcal{NF}_0$ is consistent.*

Proof: Let us suppose that $\mathcal{NF}_n$, $1 \leq n < \omega$, is non-trivial and that $\mathcal{NF}_0$ is inconsistent. Let $\alpha_1, \ldots, \alpha_n$ be a proof of a contradiction in $\mathcal{NF}_0$, where α_n is $\beta \wedge \neg\beta$. Then, if $\alpha'_1, \ldots, \alpha'_n$ are formulas obtained from the α_i as explained in the proof of the previous theorem, then this last sequence would be a derivation of $\beta' \wedge \neg^{(n)}\beta'$ in $\mathcal{NF}_n$. But in this system $(\gamma \wedge \neg_{(n)}\gamma) \rightarrow \delta$ is a valid scheme, and so $\mathcal{NF}_n$ would be trivial.▮

We can also prove the following results:

Theorem 3.1.3

1. In the hierarchy $\mathcal{NF}_n$, $1 \leq n < \omega$, every system is stronger that those that follow it.

2. If $\mathcal{NF}_1$ is non-trivial, then all $\mathcal{NF}_n$, $1 < n < \omega$, are also non-trivial.

3. $\mathcal{NF}_n$, $1 \leq n < \omega$, is inconsistent.

Let us comment on item 3. In $\mathcal{NF}_n$, Russell set, that is, the set $R =_{\text{def}} \{x : x \notin x\}$, does exist (see the next subsection). In other words, in these systems we have: $\vdash \exists y \forall x (x \in y \leftrightarrow x \notin x)$. As we will see below, it is easy to prove that $R \in R \wedge R \notin R$. So, the systems $\mathcal{NF}_n$ are inconsistent. The other two items can be proved without difficulty (see [78]).

The next step is to show that if $\mathcal{NF}_0$ is consistent, then $\mathcal{NF}_1$ is non-trivial. Therefore, due to item 2 of the previous theorem, the consistency of $\mathcal{NF}_0$ entails the non-triviality of $\mathcal{NF}_n$, $1 \leq n < \omega$.

Let us first define a system $\mathcal{NF}_1^*$ as follows: We keep only the following propositional postulates (see page 9): $\rightarrow_1 - \rightarrow_3$, $\wedge_1 - \wedge_3$, $\vee_1 - \vee_3$, and add the following new postulates: $(\rightarrow_4^*)$ (Peirce's Law) $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$, and $(\rightarrow_5^*)$ $(\neg\alpha \rightarrow \beta) \rightarrow ((\neg\alpha \rightarrow \neg\beta) \rightarrow \alpha)$, where β is not atomic. This new set of postulates provides, of course, an axiomatization for the classical propositional logic. The remaining postulates of $\mathcal{NF}_1^*$ are those of $\mathcal{NF}_1$, except for those that turn out to be redundant. For instance, since α^o is provable in $\mathcal{NF}_1^*$ when α is not atomic, it results that postulates (V) and (VI) (see page 20) are provable. So, $\mathcal{NF}_1$ is weaker than $\mathcal{NF}_1^*$.

Lemma 3.1.1 *The consistency of $\mathcal{NF}_0$ entails the non-triviality of $\mathcal{NF}_1^*$.*

Proof: Let $\mathcal{V} =_{\text{def}} \{x : x = x\}$, and let f be a map whose domain is the set of formulas of $\mathcal{NF}_1^*$ and whose range is the set of formulas of $\mathcal{NF}_0$, defined as follows:

1. $f(x = y) =_{\text{def}} x = y$
2. $f(x \in y) =_{\text{def}} x \in y$
3. $f(x \in y) =_{\text{def}} x \in y$
4. $f(x \notin y) =_{\text{def}} x \in \mathcal{V} \wedge y \in \mathcal{V}$
5. $f(\forall x \alpha) =_{\text{def}} \forall x f(\alpha)$
6. $f(\exists x \alpha) =_{\text{def}} \exists x f(\alpha)$.
7. $f(\alpha \wedge \beta) =_{\text{def}} f(\alpha) \wedge f(\beta)$
8. $f(\alpha \vee \beta) =_{\text{def}} f(\alpha) \vee f(\beta)$
9. $f(\alpha \to \beta) =_{\text{def}} f(\alpha) \to f(\beta)$

Then, using the preceding results, we can see that if α is a theorem of $\mathcal{NF}_0$, then $f(\alpha)$ is a theorem of $\mathcal{NF}_1^*$. Since the rules of inference of $\mathcal{NF}_1^*$ are valid in $\mathcal{NF}_0$, any theorem α of $\mathcal{NF}_1^*$ induces a theorem $f(\alpha)$ of $\mathcal{NF}_0$. Therefore, supposing that $\mathcal{NF}_0$ is consistent, $\mathcal{NF}_1^*$ cannot be trivial. For instance, $\emptyset \in \emptyset$ is not a theorem of $\mathcal{NF}_1^*$, since $f(\emptyset \in \emptyset) = \emptyset \in \emptyset$ is not provable in $\mathcal{NF}_0$.∎

Theorem 3.1.4 *If $\mathcal{NF}_0$ is consistent, then $\mathcal{NF}_1$ is non-trivial.*

Proof: The consistency of $\mathcal{NF}_0$ implies the non-triviality of $\mathcal{NF}_1$ because this system is weaker than $\mathcal{NF}_1^*$.∎

Theorem 3.1.5 *If $\mathcal{NF}_0$ is consistent, then all the inconsistent systems $\mathcal{NF}_n$, $1 \leq n < \omega$, are non-trivial.*

Proof: It suffices to note that the theories $\mathcal{NF}_n$, $1 \leq n < \omega$, are weaker than $\mathcal{NF}_1^*$.∎

By slightly changing the proof of Theorem 3.1.4, we can prove the following result: if $\mathcal{NF}_0$ is consistent, then the system obtained from $\mathcal{NF}_n$, $1 \leq n < \omega$, by adding axioms guaranteeing the existence of the sets of all non-k-circular sets ($k = 1, 2, \ldots$) is non-trivial. For example, the set of all non-3-circular sets is the following set: $\{x : \neg\exists y_1 \exists y_2 \exists y_3 (x \in y_1 \wedge y_1 \in y_2 \wedge y_2 \in y_3 \wedge y_3 \in x)\}$.

$\mathcal{NF}_\omega$ is like $\mathcal{NF}_1$, except that its underlying logic is $C_\omega^=$ instead of $C_1^=$.

$\mathcal{NF}_\omega$ is weaker than $\mathcal{NF}_1$, so if $\mathcal{NF}_0$ is consistent, it is non-trivial.

3.2 Zermelo-Fraenkel like systems

Starting with the Zermelo-Fraenkel system, we can introduce a new hierarchy of paraconsistent set theories $\mathcal{ZF}_n$, $1 \leq n \leq \omega$, similar to the hierarchy $\mathcal{NF}_n$, $1 \leq n \leq \omega$. Instead of ZF, we could have employed any other classical system of set theory, or type theory, as the first system in the hierarchy (see [83] and [84]). Among the several versions of ZF that can be used, perhaps the best form is Church's [64], which admits the existence of the universal set.

Let us call the latter theory $\mathcal{ZF}_0$. The language L of $\mathcal{ZF}_0$ is that of $C_1^=$, but with only one specific (binary) predicate symbol: $\in$ (membership). The syntactic notions of L are obvious adaptations of those of $C_0^=$. In addition, $\notin$ has its standard meaning, and the description symbol ι is introduced by contextual definition, following Russell. In order to state the postulates of $\mathcal{ZF}_0$, we need some definitions.

Definition 3.2.1

1. $\{x : \alpha(x)\} =_{\text{def}} \iota y \forall x (x \in y \leftrightarrow \alpha(x))$

2. $\emptyset =_{\text{def}} \{x : x \notin x\}$

3. $\mathcal{V} =_{\text{def}} \{x : x = x\}$

4. $x \subset y \leftrightarrow \forall z(z \in x \to z \in y)$

5. $\{x\} =_{\text{def}} \{y : y = x\}$

6. $\mathcal{P}(x) =_{\text{def}} \{y : y \subset x\}$

7. $\overline{x} =_{\text{def}} \{y : y \notin x\}$

8. $\exists\{x : \alpha(x)\} \leftrightarrow \exists y \forall x(x \in y \leftrightarrow \alpha(x))$

We easily define other basic notions such as $x \cup y$, $x \cap y$, $\{x, y\}$, $\langle x, y \rangle$, $\cup x$, $\cap x$, relation, function, etc.

Definition 3.2.2

1. $trans(x) \leftrightarrow \forall y(y \in x \to y \subset x)$ (transitive set)

2. $conn(x) \leftrightarrow \forall y \forall z(y \in x\ z \in x \to (z = y \vee y \in z \vee z \in y))$ (connected set)

3. $wf(x) \leftrightarrow (x \neq \emptyset \to \exists y(y \in x \wedge x \cap y = \emptyset))$ (well-founded set)

4. $ord(x) \leftrightarrow trans(x) \wedge conn(x) \wedge wf(x)$ (x is an ordinal)

5. $low(x) \leftrightarrow wf(x) \wedge$ x is equipotent to y (x is a low set)

The postulates of $\mathcal{ZF}_0$ are those of $C_0^=$ plus the following:

(P1) $\forall z(z \in x \leftrightarrow z \in y) \to x = y$ (Extensionality)

(P2) $\exists\{z : z = x \vee z = y\}$ (Pair Set)

(P3) $\exists\{z : \exists y(y \in x \wedge z \in y\}$ (Union Set)

(P4) $\exists \bigcap x$ (Intersection Set)

(P5) $\exists\{x : x$ is a finite ordinal$\}$ (Infinity)

(P6) Any formulation of the Axiom of Choice.

(P7) $low(z) \to \exists x\{x : \alpha(x) \wedge x \in z\}$ (Comprehension)

(P8) $\forall x \forall y \forall z((\alpha(x, y) \wedge \alpha(x, z) \to x = y) \wedge (\alpha(x, y) \wedge \alpha(z, y) \to x = z)) \wedge \forall y(y \in t \to \exists x \alpha(x, y)) \to (low(t) \to \exists v \forall x(x \in v \leftrightarrow \exists y(\alpha(x, y) \wedge y \in t)))$ (Replacement)

(P9) $low(x) \to \exists\{y : y \subset x\}$ (Power Set)

(P10) $\exists\{y : y \notin x\}$ (Complement)

$\mathcal{ZF}_0$ is really a strong set theory; the well-founded sets constitute a 'model' of usual Zermelo-Fraenkel set theory. Moreover, the universal set does exist in $\mathcal{ZF}_0$:

$$\vdash \exists\{x : x = x\},$$

that is, the set $\mathcal{V}$ such that $\vdash \forall x(x \in \mathcal{V})$ and, in particular, $\vdash \mathcal{V} \in \mathcal{V}$. The collection of all sets, plus $\emptyset$, $\mathcal{V}$, $\subset$, $\cup$, $\cap$ and form a complete Boolean algebra.

Theorem 3.2.1 *In $\mathcal{ZF}_0$:*

a) $\vdash x \cup \overline{x} = \mathcal{V}$ b) $\vdash x \cap \overline{x} = \emptyset$ c) $\vdash \overline{\overline{x}} = x$ d) $\vdash \overline{\mathcal{V}} = \emptyset$

e) $\vdash \bigcap \mathcal{V} = \emptyset$ f) $\vdash \bigcup \mathcal{V} = \mathcal{V}$ g) $\vdash \bigcap \emptyset = \mathcal{V}$ h) $\vdash \bigcup \emptyset = \emptyset$

i) $\vdash \neg wf(\mathcal{V})$ j) $\vdash \neg low(\mathcal{V})$ k) $\vdash \overline{\emptyset} = \mathcal{V}$

l) $x \subset y \rightarrow \overline{y} \subset \overline{x}$ m) $\vdash wf(\{\mathcal{V}\})$ n) $\vdash wf(\emptyset)$ o) $\vdash low(\emptyset)$

p) $\vdash$ *f is a function* $\wedge$ *low*(*dom*(f)) $\rightarrow$ *low*(*range*(f)), where *dom*(f) and *range*(f) stand for the domain and the range of f respectively.

Let us call α^{wf} the formula obtained from the formula α by restricting its variables by the condition $wf(\,)$. Then we have the following result, whose proof is immediate:

Theorem 3.2.2 *If α is a closed theorem of the standard ZF, then α^{wf} is a theorem of $\mathcal{ZF}_0$.*

As Church notes, it would be interesting to investigate the extension of $\mathcal{ZF}_0$ with the introduction of new postulates; for example, similar to the postulates of Quine's NF [64]. A system of this type has already been studied in [73].

Let us describe the system $\mathcal{ZF}_1$. This system is related to $\mathcal{ZF}_0$ as $C_1^=$ is related to $C_0^=$. (Recall that the latter is the standard first-order predicate calculus with identity.) So, we should have, among other things, that: (a) $\mathcal{ZF}_1$ should be partially included in $\mathcal{ZF}_0$, though the latter is also to be contained, in a certain sense, in the former. (b) $\mathcal{ZF}_1$ should be consistent, but can be used as the basis for inconsistent but non-trivial theories. In particular, in $\mathcal{ZF}_1$, we should be able to define 'inconsistent' set-theoretical structures, such as Russell set and Russell relations (see below), and other more complex structures (e.g. inconsistent arithmetics, and 'inconsistent' groups, among others).

$\mathcal{ZF}_1$ is constructed as follows: its language, called L, and the logical postulates are those of $C_1^=$. The basic set-theoretical concepts are analogous to those of $\mathcal{ZF}_0$, although the concepts involving negation give raise to two notions: one involving the weak negation ($\neg$), the other involving the strong negation ($\neg^\star$). In general, the symbols for those negations will differ only by the fact that the strong versions are starred. For instance, we have two empty sets: $\emptyset =_{\text{def}} \{x : \neg(x = x)\}$, and $\emptyset^\star =_{\text{def}} \{x : \neg^\star(x = x)\}$.

Each specific axiom or axiom scheme of $\mathcal{ZF}_0$ yields two corresponding axioms, or axiom schemas, of $\mathcal{ZF}_1$: one with the strong negation and another with the weak one. (We suppose that negation does occur essentially; otherwise, the postulate of $\mathcal{ZF}_0$ yields only one of $\mathcal{ZF}_1$, having the same syntactic form.)

It is important to note that, for example, $\mathcal{NF}_1$ is inconsistent, while $\mathcal{ZF}_1$ is apparently consistent, but non-trivial. However, it can be used as the basis for inconsistent but non-trivial theories.

The following results can be proved without difficulty. We emphasize, once again, that $\mathcal{ZF}_1$ is part of $\mathcal{ZF}_0$; but $\mathcal{ZF}_0$ is also contained, in certain sense, in $\mathcal{ZF}_1$.

Theorem 3.2.3

1. *Let* α *be a sentence of* L *and* $\alpha^\star$ *the sentence obtained from* α *by replacing* $\neg$ *by* $\neg^\star$*. Then* α *is a theorem of* $\mathcal{ZF}_0$ *iff* $\alpha^\star$ *is a theorem of* $\mathcal{ZF}_1$*.*

2. $\mathcal{ZF}_1$ *is consistent (with regard to* $\neg$ *or* $\neg^\star$*) iff* $\mathcal{ZF}_0$ *is consistent.*

3. *If* α *is a closed theorem of* $\mathcal{ZF}_0$*, then* $\forall x \forall y((x \in y)^o \wedge (x = y)^o) \rightarrow \alpha$ *is a theorem of* $\mathcal{ZF}_1$*.*

In the next subsection, we will show how $\mathcal{ZF}_1$ can be used as a paraconsistent basis for inconsistent but non-trivial theories (and, loosely speaking, for inconsistent mathematics).

As a final remark, note that in the set theories described above, we can construct semantics for C-logics, where the syntactic metalinguistic level can be considered as classical (in the sense that all the syntactic propositions are well-behaved; see [110] for further references).

3.3 Russell sets and relations

By adapting the standard notion of a mathematical structure [48], it is possible to define the concept of a paraconsistent structure [80]. However, instead of developing here a general theory of paraconsistent structures, we will consider some particular cases, and show how a certain kind of paraconsistent mathematics can be developed.

Definition 3.3.1 (Russell set) $R =_{\text{def}} \{x : x \notin x\}$

The first structure to be studied is

$$\mathfrak{R} = \langle \mathcal{V}, R \rangle, \tag{10}$$

characterized by the axiom

$$\exists R \wedge \forall x \exists \mathcal{P}(x), \tag{11}$$

where $\mathcal{V}$ is the domain (the universal set) of $\mathfrak{R}$, and R is its sole predicate; $\mathcal{P}(x)$ denotes the power set of x.

We emphasize that the following constructions are done in $\mathcal{ZF}_1$. Thus, we can make free use of the valid schemas and rules of $C_1^=$.

Theorem 3.3.1 $\vdash R \in R \wedge R \notin R$

Proof: Given the definition of R, $x \in R \leftrightarrow x \notin x$. Hence, replacing x for R, we get $R \in R \leftrightarrow R \notin R$. However, if $R \in R$, it follows that $R \notin R$, and if $R \notin R$, we obtain $R \notin R$. Therefore, by excluded middle, $R \notin R$. Similarly, we prove that $R \in R$ by assuming that $R \notin R$. ▮

Theorem 3.3.2 $\vdash y \in \{x\} \leftrightarrow y = x$

Proof: In $\mathcal{ZF}_1$, we have that: $\exists\{x\}$. Therefore, $\vdash y \in \{x\} \leftrightarrow y = x$, by the definition of $\{x\}$. ▮

Theorem 3.3.3 $\vdash x \in R \rightarrow \{x\} \in R$

Proof: Either $\{x\} \notin \{x\}$ or $\{x\} \in \{x\}$. In the first case, $\{x\} \in R$, by the definition of R. In the second case, $\{x\} = x$, and given the hypothesis, $\{x\} \in R$.▮

Theorem 3.3.4 $\vdash x, y \in R \rightarrow \{x, y\} \in R$

Proof: Either $\{x,y\} \notin \{x,y\}$ or $\{x,y\} \in \{x,y\}$. In the first case, $\{x,y\} \in R$, by the definition of R. In the second case, either $\{x,y\} = x$ or $\{x,y\} = y$, and given the hypothesis, $\{x,y\} \in R$.▮

Theorem 3.3.5 $\vdash \{\{x, R\}\} \in R$

Proof: Either $\{\{x,R\}\} \in \{\{x,R\}\}$ or $\{\{x,R\}\} \notin \{\{x,R\}\}$. In the second case, it is immediate that $\{\{x,R\}\} \in R$. In the first case, by Theorem 3.3.2, it follows that $\{x,R\} = \{\{x,R\}\}$. Therefore, $x = R = \{x,R\}$, and given that $R \in R$, by Theorem 3.3.4, $\{x,R\} \in R$. Thus, by Theorem 3.3.3, $\{\{x,R\}\} \in R$.▮

Theorem 3.3.6 (Arruda and Batens, 1982) $\bigcup R = \mathcal{V}$.

Proof: It suffices to prove that, for every x, $x \in \bigcup R$. Let us suppose that (i) $\{x,R\} \notin \{x,R\}$. Hence, $\{x,R\} \in R$ and, by the definition of the union set, $x \in \bigcup R$. On the other hand, if (ii) $\{x,R\} \in \{x,R\}$, then either $\{x,R\} = x$ or $\{x,R\} = R$. In the second case, it follows that $x \in \bigcup R$. If $\{x,R\} = x$, we have that $\{\{x,R\}\} = \{x\}$, and given that $\{\{x,R\}\} \in R$ (Theorem 3.3.5), it follows that $\{x\} \in R$. Hence, $x \in \bigcup R$.▮

This last theorem shows that a set theory with Russell set has, in general, a universal class.

Theorem 3.3.7 (Arruda) $\ldots \subset \mathcal{P}(\mathcal{P}(R)) \subset \mathcal{P}(R) \subset R$.

Proof: If $x \in \mathcal{P}(R)$, then $x \subset R$. Now, either $x \notin x$ or $x \in x$. If $x \notin x$, then $x \in R$; if $x \in x$, given that $x \subset R$, it follows that $x \in R$. Therefore, $\mathcal{P}(R) \subset R$. Furthermore, if $x \in \mathcal{P}(\mathcal{P}(R))$, then $x \subset \mathcal{P}(R)$, and by the previous result, $x \subset R$; hence $x \in \mathcal{P}(R)$. Thus, $\mathcal{P}(\mathcal{P}(R)) \subset \mathcal{P}(R) \subset R$. It is now easy to complete the proof.▮

Theorem 3.3.8 *In $\mathcal{ZF}_1$ plus the axiom (11):*

1. $\vdash \emptyset \in R, \{\emptyset\} \in R, \{\{\emptyset\}\} \in R \ldots$
2. $\vdash \exists x (x \notin R)$
3. $\vdash \mathcal{V} \notin R$
4. $\vdash x, y \in R \rightarrow \langle x, y \rangle \in R$
5. $\vdash \mathcal{V} = \mathcal{P}(\mathcal{V})$
6. $\vdash x \subset R \rightarrow x \in R$
7. $\vdash R \cup \overline{R} = \bigcup R$
8. $\vdash R \times R \subset R$
9. $\vdash \neg^{\star} \exists \{x : \neg^{\star}(x \in x)\}$

10 $\vdash \exists R \rightarrow \exists x(x \text{ is infinite})$

Given Theorem 3.3.5, it is possible to demonstrate that R is, as it were, an 'internal model' of the set theory in which we are working. Moreover, given that $\bigcup R = \mathcal{V}$, it follows that the existence of R implies the existence of infinite sets. The properties of R are by no means arbitrary. For example, it does not seem possible to prove that $(R \in R)^o$. Moreover, it is not possible to prove everything with regard to R without also proving, at the same time, that some classical, standard set theories are inconsistent (see [69] and [78]). Finally, $\mathcal{ZF}_1$ with the extra axiom $\exists R$ is non-trivial iff $\mathcal{ZF}_0$ is consistent.

3.3.1 Russell relations

Definition 3.3.2 (Russell relations) *Let* $n = 1, 2, \ldots$, *and* $0 < i \leq n$. *Then:*

$$R_{n,i} =_{\text{def}} \{\langle x_1, \ldots, x_n\rangle : \langle x_1, \ldots, x_n\rangle \notin x_i\}.$$

By convention, we have: $\langle x\rangle = x$.

In particular, $R_{1,1} = R$. With $R_{n,i}$, assuming appropriate postulates, we can build structures similar to $\mathfrak{R}$ above. We can also prove the following results:

Theorem 3.3.9

1. $\vdash R_{n,i} \in R_{n,i} \wedge R_{n,i} \notin R_{n,i}$
2. $\vdash \mathcal{V} \times \cdots \times \mathcal{V} = \bigcup R_{n,i}$, where the product on the left has n terms. In particular, $\vdash \mathcal{V} \times \mathcal{V} \subset \cup \cup R_{2,1}$

Theorem 3.3.10

1. $\vdash \forall t(\langle \emptyset, t\rangle \in R_{2,1})$
2. $\vdash \cup \cup R_{2,1} = \mathcal{V}$

Proof: With regard to 1, by hypothesis, $\exists R_{2,1}$. Then, $\langle \emptyset, t\rangle \in R_{2,1} \leftrightarrow \langle \emptyset, t\rangle \notin \emptyset$. With regard to 2, since $\{\{\emptyset\}, \{\emptyset, t\}\} = \langle \emptyset, t\rangle \in R_{2,1}$, it follows that $\{\emptyset, t\} \in \cup R_{2,1}$ and $t \in \cup \cup R_{2,1}$.▌

Given the consistency of $\mathcal{NF}_0$ (see the previous subsection), if we add to $\mathcal{NF}_n$, $1 \leq n < \omega$, new postulates guaranteeing the existence of Russell relations, the resulting system is non-trivial.

3.4 Paraconsistent Boolean algebra

In various paraconsistent set theories, even in classical set theories such as Zermelo-Fraenkel, it is possible to consider intuitively a set as an ordered pair, in the classical sense, of sets that are part of a universal class $\mathcal{V}$. Thus, a set X is a pair $\langle X_1, X_2\rangle$, where:

(1) $x \in X$ iff $x \in X_1$

(2) $x \notin X$ iff $x \in X_2$

(3) $x \in X$ and $x \notin X$ is equivalent to $x \in X_1$ and $x \in X_2$

Given that the principle of excluded middle holds in certain paraconsistent set theories, it should be the case that $X_1 \cup X_2 = \mathcal{V}$. If $X_1 \cap X_2 = \emptyset$, a classical set is obtained. Let us consider then the collection of the sets just constructed on $\mathcal{V}$, which will be denoted by $\mathfrak{V}$. An element of $\mathfrak{V}$ is called a paraconsistent set, or a p-set. In what follows, we outline an algebra of p-sets $\mathfrak{V}$. We suppose that the p-sets are embedded in a classical set theory, for instance, ZF.

Suppose that $X = \langle X_1, X_2 \rangle$ and $Y = \langle Y_1, Y_2 \rangle$. In this case:

Definition 3.4.1

(i) $X \sqcup Y =_{\text{def}} \langle X_1 \cup Y_1, X_2 \cap Y_2 \rangle$

(ii) $X \sqcap Y =_{\text{def}} \langle X_1 \cap Y_1, X_2 \cup Y_2 \rangle$

(iii) $\mathbf{1} =_{\text{def}} \langle \mathcal{V}, \emptyset \rangle$

(iv) $\mathbf{0} =_{\text{def}} \langle \emptyset, \mathcal{V} \rangle$

(v) $\overline{X} =_{\text{def}} \langle X_2, X_1 \rangle$

(vi) $X \sqsubset Y =_{\text{def}} X_1 \subset Y_1 \wedge Y_2 \subset X_2$

Theorem 3.4.1

$\vdash X \sqcup X = X$

$\vdash X \sqcup Y = Y \sqcup X$

$\vdash (X \sqcup Y) \sqcup Z = X \sqcup (Y \sqcup Z)$

$\vdash \boldsymbol{1} \sqcup X = \boldsymbol{1}$

$\vdash \boldsymbol{0} \sqcup X = X$

$\vdash X \sqsubset X$

$\vdash X \sqsubset Y \wedge Y \sqsubset X \to X = Y$

$\vdash X \sqsubset X \sqcup \overline{X}$

$\vdash X \sqcap X = X$

$\vdash X \sqcap Y = Y \sqcap X$

$\vdash (X \sqcap Y) \sqcap Z = X \sqcap (Y \sqcap Z)$

$\vdash \boldsymbol{1} \sqcap X = X$

$\vdash X \sqcup \overline{X} \sqsubset \boldsymbol{1}$

$\vdash \boldsymbol{0} \sqsubset X$

$\vdash X \sqsubset Y \wedge Y \sqsubset Z \to X \sqsubset Z$

$\vdash X \sqcap \overline{X} \sqsubset X$

$\vdash \overline{\overline{X}} = X$

$\overline{\boldsymbol{1}} = \boldsymbol{0}$

$\vdash \overline{\boldsymbol{0}} = \boldsymbol{1}$

$\vdash \boldsymbol{0} \sqcap X = \boldsymbol{0}$

$\vdash \boldsymbol{0} \sqsubset X \sqcap \overline{X}$

$\vdash X \sqsubset \boldsymbol{1}$

Definition 3.4.2 *The structure $\mathfrak{P} = \langle \mathfrak{V}, \sqcup, \sqcap, \overline{}, \boldsymbol{1}, \boldsymbol{0} \rangle$ is called a paraconsistent Boolean algebra.*

Using this structure, it is possible to formalize several paraconsistent patterns of reasoning, just as with classical Boolean algebras, we can put in algebraic terms various classical inferences. We can also verify that the paraconsistent logical mechanism considered here does not exclude classical logic, but extends it in a certain sense – though under another point of view, it can be embedded into classical logical structures. Of course, this point is valid for particular categories of paraconsistent structures. It corroborates the fact that both paraconsistent logic and paraconsistent mathematics do not destroy either traditional logic or standard mathematics. Rather, the paraconsistent approach complements the classical developments of logic and mathematics, and in certain cases, extends them.

The structure of the paraconsistent Boolean algebra clearly is richer than the classical one. Thus, for instance, one can introduce two operators, π_1 and π_2, such that, given a p-set X, $\pi_1(X) = X_1$ and $\pi_2(X) = X_2$, where X_1 and X_2 are in another Boolean algebra, the classical algebra of the subsets of $\mathcal{V}$ etc.

When the structure $\mathfrak{P}$, in the above definition, is such that, for every $X = \langle X_1, X_2\rangle$, it is the case that $X_1 \cap X_2 = \emptyset$, one obtains a Boolean algebra that essentially is the usual algebra of the subsets of $\mathcal{V}$.

In this way, it is possible to construct a general theory of paraconsistent structures (algebraic, topological, of order etc.), obtaining in this way a generalization of the traditional theory of structures, such as Bourbaki's. Moreover, as we will see below, paraconsistent structures, such as those described in this section, have been applied to several areas, such as computer science, artificial intelligence and logic programming (see, e.g., [231], [38], [39], and [156]). [13] This provides a significant motivation for their study.

3.5 Paraconsistent mathematics

In this subsection, we outline a paraconsistent formulation of elementary differential and integral calculus, in which what we will call l'Hospital principle is true:

> 'Two finite quantities that differ by an infinitely small quantity are equal.' [14]

The paraconsistent nature of this principle is clear, even though we are not trying to advance an exegesis of l'Hospital's work (see [175] and [219]). To begin with, we describe two (classical) algebraic structures $\mathcal{A}$ and $\mathcal{A}^\star$.

The ring $\mathcal{A}$ The ring $\mathcal{A}$ is described as follows. Let $\mathbb{R}$ denote the field of real numbers and a an element of a fixed open interval I ($I \subset \mathbb{R}$). An *infinitesimal variable* is a real valued function defined on I that has limit zero in a (we can use right or left limits). The expression 'infinitesimal variable' is here employed inspired by the terminology of the classical French treatises, such as those by Picard and Gousart, as well as by Cauchy. The elements of $\mathcal{A}$ are ordered pairs $\langle r, f\rangle$, where $r \in \mathbb{R}$ and f is an infinitesimal variable.

If $\langle r, f\rangle$ and $\langle s, g\rangle$ belong to $\mathcal{A}$, then $\langle r, f\rangle = \langle s, g\rangle$ if $r = s$ and $f = g$. Addition is defined as follows: $\langle r, f\rangle + \langle s, g\rangle =_{\text{def}} \langle r + s, f + g\rangle$. By an abuse of language, we take $0 = \langle 0, 0\rangle$, where the second zero in $\langle 0, 0\rangle$ is the identically zero function in I. We also take $-\langle r, f\rangle =_{\text{def}} \langle -r, -f\rangle$, and $\langle r, f\rangle - \langle s, g\rangle =_{\text{def}} \langle r, f\rangle + (-\langle s, g\rangle) = \langle r - s, f - g\rangle$.

The following additional definitions are needed: $\langle r, f\rangle \cdot \langle s, g\rangle =_{\text{def}} \langle rs, rg + fs + fg\rangle$ and $1 =_{\text{def}} \langle 1, 0\rangle$. Furthermore, if $r \neq 0$, we put $\langle r, f\rangle^{-1} =_{\text{def}} \langle r^{-1}, -f/r(f + r)\rangle$ and $\langle s, g\rangle \div \langle r, f\rangle =_{\text{def}} \langle s, g\rangle \cdot \langle r, f\rangle^{-1}$.

The elements of $\mathcal{A}$ with the operation + constitute a commutative group, and with ×, a commutative semi-group with unity. Moreover, multiplication is distributive in relation to addition. Therefore, $\mathcal{A}$ is a commutative ring with unity. The elements of $\mathcal{A}$

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[13] It is worth noting that paraconsistent structures have also been employed in quantum mechanics by Dalla Chiara and Giuntini (see [118], [119], and [120]).

[14] In this context, it is worth also giving a quotation attributed to Johann Bernoulli (1667-1748), according to which 'a quantity that is increased or decreased by an infinitely small quantity is neither increased nor decreased'.

are called *hyper-reals*. If we identify $\langle r, 0\rangle$ with r, then the field $\mathbb{R}$ is contained in $\mathcal{A}$ as a sub-ring. We write $\langle r, f\rangle$ as $r + \langle 0, f\rangle$. A pair such as $\langle 0, f\rangle$ is called an *infinitesimal*, and will be usually denoted by small Greek letters. Any element of $\mathcal{A}$, $\langle r, f\rangle$, then, is the sum of a standard real and an infinitesimal: $\langle r, f\rangle = r + \varepsilon$, where $\varepsilon = \langle 0, f\rangle$.

Division can be extended to infinitesimals. In fact, let us suppose that $\kappa = \langle 0, f\rangle$, $\lambda = \langle 0, g\rangle$, and $\lim f/g = r$ in $\mathcal{A}$. Also, let $\kappa/\lambda = \langle r, f/g - r\rangle$, where the variable g is supposed not to assume the value 0. Therefore, making $h = f/g - r$, it is easy to check that the infinitesimal $\varepsilon = \langle 0, h\rangle$ is such that $\kappa = \lambda \cdot \varepsilon$. When $\lim f/g = \infty$, or does not exist, the quotient κ/λ is not defined.

We introduce in $\mathcal{A}$ a relation of inequality, $<$, so that $\langle r, f\rangle < \langle s, g\rangle$ if $r < s$, or in case $r = s$, if $f < g$ (the variable f is less than the variable g in all points of I).

The set of all hyper-reals of the form $\langle r, f\rangle$ is called *the monad* of $r \in \mathbb{R}$. The *order* of an infinitesimal ε in relation to another infinitesimal κ is defined without difficulty. Given a function $f : \mathbb{R} \mapsto \mathbb{R}$, it can be extended so that $dom(f) = \mathcal{A}$, under the hypothesis that $\lim_{x\to t} f(x)$ exists and is finite for any $t \in \mathbb{R}$. We take $f(r + \varepsilon) = b + \delta$, where $b = \lim_{x\to r} f(x)$ and δ is an infinitesimal obviously defined.

The basic concepts of the differential calculus can then be defined in terms of infinitesimals. For example, the expression:

$$\lim_{x\to r} f(x) = b$$

means that $f(r + \varepsilon) = b + \delta$, for every infinitesimal ε (δ is also an infinitesimal), where $r \in \mathbb{R}$. Similarly, the derivative of f is defined as follows:

$$f(r + \varepsilon) - f(r) = f'(r) \cdot \varepsilon + \delta,$$

for any infinitesimal ε, δ being an infinitesimal of order higher than that of ε. The properties of limits, derivatives etc. are easily proved from the properties of infinitesimals.

The quasi-ring $\mathcal{A}^\star$ The quasi-ring $\mathcal{A}^\star$ is obtained from $\mathcal{A}$ by the introduction of infinite 'numbers'. An *infinite variable* is a standard real-variable function, defined in I, and divergent at $a \in I$.

The pair $\langle v, 0\rangle$ is called an infinite *hyper-real* number when v is an infinite variable. The (finite) elements of $\mathcal{A}$ and the infinite numbers form $\mathcal{A}^\star$. Two infinite numbers $\langle v, 0\rangle$ and $\langle u, 0\rangle$ are equal if $v = u$. The operations of $\mathcal{A}^\star$ are defined as in $\mathcal{A}$, but extended as follows:

Addition

1. If $\langle v, 0\rangle$ is infinite and $\langle k, f\rangle$ is finite, then $\langle v, 0\rangle + \langle k, f\rangle = \langle k, f\rangle + \langle v, 0\rangle = \langle v + k, 0\rangle$.
2. If $\langle v, 0\rangle$ and $\langle u, 0\rangle$ are both infinite, then $\langle v, 0\rangle + \langle u, 0\rangle = \langle u, 0\rangle + \langle v, 0\rangle = \langle v + u, 0\rangle$, whenever $\lim(v + u) = \infty$; $\langle v, 0\rangle + \langle u, 0\rangle = \langle u, 0\rangle + \langle v, 0\rangle = \langle k, f\rangle$ when $\lim(v + u) = k$ (standard real), where $f = k - (u + v)$. Otherwise, $\langle v, 0\rangle + \langle u, 0\rangle$ and $\langle u, 0\rangle + \langle v, 0\rangle$ are not defined.

Subtraction By definition, the opposite of an infinite $\langle v, 0\rangle$ is $\langle -v, 0\rangle$. Then, the difference of two elements of $\mathcal{A}^\star$ is defined as the sum of the first with the opposite of the second.

Multiplication

1. If $\langle v, 0\rangle$ and $\langle u, 0\rangle$ are infinite, then $\langle v, 0\rangle \cdot \langle u, 0\rangle = \langle u, 0\rangle \cdot \langle v, 0\rangle = \langle uv, 0\rangle$.
2. If $\langle v, 0\rangle$ is infinite and $\langle k, f\rangle$ is finite, but not an infinitesimal, then $\langle v, 0\rangle \cdot \langle k, f\rangle = \langle k, f\rangle \cdot \langle v, 0\rangle = \langle vk, 0\rangle$
3. If $\langle v, 0\rangle$ is infinite and $\langle 0, f\rangle$ is an infinitesimal, then $\langle v, 0\rangle \cdot \langle 0, f\rangle = \langle 0, f\rangle \cdot \langle v, 0\rangle = \langle vf, 0\rangle$, when $\lim(vf) = \infty$; if $\lim(vf) = 0$, the product is equal to $\langle 0, vf\rangle$. Otherwise, the product is not defined.

Division Let $\langle v, 0\rangle$ be an infinite number satisfying obvious conditions; $\langle v, 0\rangle^{-1}$ is the infinitesimal $\langle 0, v^{-1}\rangle$. If $\langle 0, f\rangle$ is an infinitesimal satisfying appropriate conditions, then $\langle 0, f\rangle^{-1}$ is the infinite $\langle f^{-1}, 0\rangle$. The quotient of two elements of $\mathcal{A}^\star$ is the product of the first by the inverse of the second.

The relation $<$ can be extended to $\mathcal{A}^\star$ without difficulty. (However, $\mathcal{A}$ and $\mathcal{A}^\star$ are not Archimedean structures.) In terms of infinitesimals and infinities, we can express in $\mathcal{A}^\star$ the basic ideas and results of the infinitesimal calculus (for analogous views, see [226], [168], and [169]). In particular, we can get something like de l'Hospital's principle on curves, according to which a smooth curve may be analyzed into an infinite number of infinitesimal straight lines ([175], and [219]).

The classical theory of infinitesimals and infinite quantities, as well as Du Bois Reymond's theory of orders of magnitude concerning the asymptotic behavior of functions (see [47, Note II]), are translatable in the language of $\mathcal{A}^\star$. The same is true in connection with other topics of pure and applied mathematics, such as, the theory of differential equations and Fourier series. Loosely speaking, $\mathcal{A}^\star$ is a model of a theory with infinitesimals and infinite quantities. It is also worth noting that natural suppositions about the orders of infinitesimals and infinities give raise to propositions that are undecidable in Zermelo-Fraenkel set theory (see [127]).

To describe a paraconsistent model for the differential and integral calculus, we start with the language L in which we can treat the central notions of this calculus. L is essentially Manin's L_2Real [178, p. 109], conveniently extended by the introduction of names, in the sense of Shoenfield (see [228]), for all elements of $\mathcal{A}$ (or of $\mathcal{A}^\star$). So, L is composed of the following primitive symbols: (1) individual variables; (2) variables for functions of one variable; (3) individual constants: the names of the elements of $\mathcal{A}$ (or of $\mathcal{A}^\star$); (4) the symbols $+$ and $\times$ for binary operations; (5) two binary relation symbols: $\equiv$ and $<$; (6) the connectives: $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$ and $\neg$; and (7) the quantifiers: $\forall$ and $\exists$; (8) parentheses.

Terms of L: (a) All individual variables, individual constants, and function variables are terms. (b) If F is a function variable and t is a term, then $F(t)$ is a term. (c) If t_1 and t_2 are terms, so are $t_1 + t_2$ and $t_1 \times t_2$.

Formulas of L: If t_1 and t_2 are terms, then $t_1 \equiv t_2$ and $t_1 < t_2$ are atomic formulas. The remaining formulas are defined as usual, but quantification over function variables is allowed. The common syntactic notions such as those of bound and free variables, sentence etc. are defined in the usual way.

The following clauses define when v is a paraconsistent valuation of L:

1. Names denote the corresponding elements of $\mathcal{A}$ (or of $\mathcal{A}^\star$).

2. Let $t_1 < t_2$ be an atomic sentence. In this case, $v(t_1 < t_2) = 1$ if $t_1 < t_2$ is true in $\mathcal{A}$ (or in $\mathcal{A}^\star$). Otherwise, $v(t_1 < t_2) = 0$.

3. $v(t_1 \equiv t_2) = 1$ iff $t_1 - t_2$ is infinitesimal with respect to ε (ε is an infinitesimal).

4. Let $t_1 \equiv t_2$ be an atomic sentence. In this case, $v(t_1 \not\equiv t_2) = v(\neg(t_1 \equiv t_2)) = 1$ if $t_1 \neq t_2$ in $\mathcal{A}$ (or in $\mathcal{A}^\star$); otherwise, $v(t_1 \not\equiv t_2) = 0$.

5. Similarly, we define the value of v for any sentence of L, replacing $\equiv$ and $\not\equiv$ in the sentence by convenient symbolic combinations, respectively as in 3 and 4.

If a sentence F is undecidable in the usual set theory plus the axioms of $\mathcal{A}$ (or of $\mathcal{A}^\star$), the value of F is chosen arbitrarily, in such a way that v be the characteristic function of a maximal non-trivial set of sentences. Therefore, v is really a valuation in L.

It is immediate to verify that we may have, for some terms t_1 and t_2, that $v(t_1 \equiv t_2) = v(t_1 \not\equiv t_2) = 1$. In fact, v constitutes a paraconsistent valuation and determines a *model* of the infinitesimal calculus in which de l'Hospital's principle about finite quantities that differ by an infinitely small increment holds. Moreover, in the case of $\mathcal{A}^\star$, the model is such that even de l'Hospital's second principle, on smooth curves, happens to be valid (when conveniently interpreted).

The method sketched here to obtain v is analogous to those of Chris Mortensen (see [191]), and of synthetic differential geometry (see [41] and [42]). For the treatment of functions of several variables, we have to strengthen L with the introduction of functional variables (see [225]).

In summary, we may say that v is a paraconsistent structure, which we will call the *de l'Hospital structure*. On the paraconsistent version of the infinitesimal calculus outlined here, see also [110]. Further developments could start with the paradoxes of Burali-Forti and of Cantor, which naturally motivate the definition of interesting paraconsistent structures. The development of paraconsistent arithmetics, in turn, offers no difficulty at all.

4 Jaśkowski's Logic

4.1 Jaśkowski's Discussive Logic

As noted above, following a suggestion by J. Łukasiewicz, Stanislaw Jaśkowski (1906-1965) was the first logician to construct a system of paraconsistent propositional calculus (see [149], [150], [151], and [89]). Jaśkowski motivated his *discussive* logic (sometimes also referred to as *discursive* logic) by the need for addressing three issues: (i) to systematize theories that contain contradictions, such as dialectics; (ii) to study theories where there are contradictions originated by vagueness, and (iii) to investigate empirical theories whose postulates or basic assumptions are contradictory (see [15] and [16]).

Later, da Costa and Dubikajtis extended the discussive propositional calculus to first- and higher-order predicate calculi (see [90] and [91]; see also [160], [161], and [162]). Recently, discussive logic has been applied to the theory of pragmatic truth (as we will discuss below), to the foundations of physics (see [89] and also below), and in the philosophy of science (for a general account, see [97]). In this section, we will sketch the main ideas related to Jaśkowski's discussive logic.

Let us call $\mathcal{J}$ the discussive propositional calculus whose language $\mathcal{L}$ and notations are those one the modal system S5. We pose: $\Diamond\Gamma =_{\mathsf{def}} \{\Diamond\alpha : \alpha \in \Gamma\}$. So, $\mathcal{J}$ can be semantically defined as

$$\Gamma \models_{\mathcal{J}} \alpha \text{ iff } \Diamond\Gamma \models_{S5} \Diamond\alpha,$$

where the notation has an obvious meaning. It is immediate that

Theorem 4.1.1

(1) $\models_{\mathcal{J}} \alpha$ *iff* $\models_{S5} \Diamond\alpha$

(2) $\Gamma \models_{\mathcal{J}} \alpha$ *iff there are* $\gamma_1, \ldots, \gamma_n$ in Γ *such that* $\models_{S5} \Diamond\gamma_1 \wedge \ldots \wedge \Diamond\gamma_n \to \Diamond\alpha$

Corolary 4.1.1 $\Gamma \models_{\mathcal{J}} \alpha$ *iff there is a finite set* $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ *such that* $\{\gamma_1, \ldots, \gamma_n\} \models_{\mathcal{J}} \alpha$.

Corolary 4.1.2 *If* $\models_{S5} \alpha$*, then* $\models_{\mathcal{J}} \alpha$.

Proof: It is enough to note that if $\models_{S5} \alpha$, then $\models_{\mathcal{J}} \Diamond\alpha$.▮

Given the definitions and results above, we can see that $\mathcal{J}$ accomplishes Jaśkowski's main intensions.

$\mathcal{J}$ has several axiomatizations (see [75] and [91]); the one presented here was introduced in [89]. The postulates are:

(J1) If α is an axiom of S5, then $\Box\alpha$.

(J2) $\Box\alpha, \Box(\alpha \to \beta) / \Box\beta$

(J3) $\Box\alpha / \alpha$

(J4) $\Diamond\alpha / \alpha$

(J5) $\Box\alpha / \Box\Box\alpha$

Lemma 4.1.1 *If* $\vdash_{\mathcal{J}} \alpha$ *means that* α *is provable in* $\mathcal{J}$*, then: if* $\vdash_{\mathcal{J}} \alpha$*, then* $\models_{S5} \Diamond\alpha$.

Proof: By induction on the length of a given derivation of α in $\mathcal{J}$.▮

Lemma 4.1.2 *If* $\models_{S5} \alpha$*, then* $\vdash_{\mathcal{J}} \Box\alpha$.

Proof: By induction on the length of a given derivation of α in S5.▮

Theorem 4.1.2 $\vdash_{\mathcal{J}} \alpha$ *iff* $\models_{\mathcal{J}} \alpha$.

Proof: If $\vdash_{\mathcal{J}} \alpha$, then by Lemma 4.1.1, $\models_{S5} \Diamond\alpha$. So, by definition, $\models_{\mathcal{J}} \alpha$. Conversely, if $\models_{\mathcal{J}} \alpha$, then by definition $\models_{S5} \Diamond\alpha$. So, by Lemma 4.1.2, $\vdash_{\mathcal{J}} \Box\Diamond\alpha$. By postulate (J3), $\vdash_{\mathcal{J}} \Diamond\alpha$, and by postulate (J4), $\vdash_{\mathcal{J}} \alpha$.▮

Definition 4.1.1 *We write* $\Gamma \vdash_{\mathcal{J}} \alpha$ *iff there are* $\gamma_1, \ldots, \gamma_n$ *such that* $\vdash_{\mathcal{J}} \Diamond\gamma_1 \wedge \ldots \wedge \Diamond\gamma_n \to \Diamond\alpha$.

Theorem 4.1.3 $\Gamma \vdash_{\mathcal{J}} \alpha$ *iff* $\Gamma \models_{\mathcal{J}} \alpha$.

Proof: Immediate consequence of the definition above and Theorem 4.1.2.▮

Theorem 4.1.4 *Modus Ponens, that is, the rule* $\alpha, \alpha \to \beta / \beta$, *is not valid in* $\mathcal{J}$.

Proof: The uniform predicate calculus $\mathcal{U}$ is a subcalculus of the monadic calculus; it is, the first-order predicate calculus dealing only with unary predicates in which there is only one individual variable, say x (see [63]). There is an obvious bijection between the set of formulas of $\mathcal{U}$ and the language of S5: given a formula α of $\mathcal{U}$, we obtain the corresponding formula α' of the language of S5 by replacing any subformula $P_i(x)$ of α by p_i, and any universal quantification $\forall x$ by $\Box$. We can show that if α is a formula of $\mathcal{U}$ and α' is its corresponding formula in S5, then $\models_{\mathcal{U}} \alpha$ iff $\models_{S5} \alpha'$ (see [89]). To prove the theorem, it is enough to note that $\exists x\alpha(x)$ and $\exists x(\alpha(x) \to \beta(x))$ do not imply that $\exists x\beta(x)$ in $\mathcal{U}$.▮

Theorem 4.1.5 *(a) The rules* $\alpha, \beta / \alpha \wedge \beta$ *and* $\alpha, \neg\alpha / \beta$ *are not valid in* $\mathcal{J}$. *(b) The deduction theorem* $\Gamma, \alpha \vdash_{\mathcal{J}} \beta \Rightarrow \Gamma \vdash_{\mathcal{J}} \alpha \to \beta$ *is not true in* $\mathcal{J}$.

These results show that $\mathcal{J}$ can be used to accommodate inconsistent sets of premisses while avoiding triviality, that is, it is paraconsistent.

Definition 4.1.2

(1) [Discussive Implication] $\alpha \to_d \beta =_{\text{def}} \Diamond\alpha \to \beta$

(2) [Discussive Conjunction] $\alpha \wedge_d \beta =_{\text{def}} \Diamond\alpha \wedge \beta$

(3) [Discussive Equivalence] $\alpha \leftrightarrow_d \beta =_{\text{def}} (\alpha \to_d \beta) \wedge_d (\beta \to_d \alpha)$

(4) [Impossibility] $\nabla\alpha =_{\text{def}} \neg\Diamond\alpha$

Theorem 4.1.6 *The connectives* $\to_d$, $\vee$, $\wedge_d$, $\leftrightarrow_d$ *and* $\leftrightarrow_d$ *have all the classic properties of* $\to$, $\vee$, $\wedge$, $\leftrightarrow$ *and* $\neg$ *respectively.*

This shows that the classical propositional calculus is naturally embedded in $\mathcal{J}$. However, as the next theorem shows, several rules and formulas do not hold in this calculus.

Theorem 4.1.7 *The following formulas and rules are not valid in* $\mathcal{J}$.

(1) $\alpha \to_d (\beta \to_d \alpha \wedge \beta)$

(2) $\alpha \rightarrow_d (\neg\alpha \rightarrow_d \beta)$

(3) $(\alpha \wedge \beta \rightarrow_d \gamma) \rightarrow_d (\alpha \rightarrow_d (\beta \rightarrow_d \gamma))$

(4) $\Gamma, \alpha \models_{\mathcal{J}} \beta$ and $\Gamma, \alpha \models_{\mathcal{J}} \neg\beta \Rightarrow \Gamma \models_{\mathcal{J}} \neg\alpha$

(5) $(\alpha \leftrightarrow_d \neg\alpha) \rightarrow_d \beta$

(6) $(\alpha \rightarrow_d \neg\alpha) \rightarrow_d \neg\beta$

Theorem 4.1.8 (1) $\mathcal{J}$ is decidable.

(2) $\mathcal{J}$ has no finite characteristic matrix, but has the finite model property.

Proof: With regard to (1), the decidability of $\mathcal{J}$ follows from S5's decidability. In fact, if α is a formula of $\mathcal{J}$, use Definition 4.1.2 to obtain a formula β of S5 by eliminating the discussive connectives. Now, since β is a theorem of $\mathcal{J}$ iff $\Diamond\beta$ is a theorem of S5, use a decision procedure for S5 to verify whether $\Diamond\beta$ is a theorem of S5. With regard to (2), it follows from the fact that S5 has the finite model property, even though it has no finite characteristic matrix (for details, see [75]).▮

$\mathcal{J}$ has (at least) two equivalent semantics. One based on Kripke structures, the other based on the notion of Hanle algebra. A Hanle algebra is a structure $\mathcal{H} = \langle A, -, \wedge, \star \rangle$, where $\langle A, -, \wedge \rangle$ is a Boolean algebra and $\star$ is an unary operator over A such that: $\star 1 = 1$ and $\star x = 0$, for all $x \neq 0$, where 0 and 1 are, respectively, the first and the last elements of the Boolean algebra. We can prove that the following sentences are equivalent (where α is a formula of S5): (i) α is a theorem of S5; (ii) α is valid in every Hanle algebra, and (iii) α is valid in every finite Hanle algebra (see [89]). From these statements, it is easy to derive a semantics for $\mathcal{J}$. Nevertheless, in what follows, we will make reference only to the semantics based on Kripke structures.

A Kripke semantics for $\mathcal{J}$ can be constructed by the same steps we take to build a model theory for S5, and the corresponding results can be proved.

Definition 4.1.3 *Let Γ be a set of formulas of the language of $\mathcal{J}$. Then:*

(1) $\overline{\Gamma} =_{\text{def}} \{\alpha : \Gamma \vdash_{\mathcal{J}} \alpha\}$

(2) *If $\overline{\Gamma}$ is the set of all formulas, then Γ is trivial; otherwise, it is non-trivial.*

(3) *If there is a formula α such that $\Gamma \vdash_{\mathcal{J}} \alpha$ and $\Gamma \vdash_{\mathcal{J}} \neg\alpha$, then Γ is inconsistent; otherwise, it is consistent.*

(4) *If there is a formula α such that $\Gamma \vdash_{\mathcal{J}} \alpha$ and $\Gamma \vdash_{\mathcal{J}} \nabla\alpha$, then Γ is strongly inconsistent.*

Theorem 4.1.9 *There are inconsistent, but non-trivial sets of formulas.*

Proof: If p is a propositional variable, then $\{p, \neg p\}$ is inconsistent, but non-trivial.▮

A Kripke structure is an ordered pair $K = \langle W, \upsilon \rangle$, where W is a non-empty set whose elements are called *worlds*, and υ is a mapping $\upsilon : W \times P \mapsto \{0, 1\}$, where P is the set of propositional variables of the language of $\mathcal{J}$. As usual, if $\upsilon(w, p) = 1 (= 0)$, we say that p is *true* (*false*) in the world w.

For brevity's sake, let us take $\vee$, $\neg$, and $\Box$ as the primitive connectives of our language. If α is a formula, then we say that K, w *force* α, and write $K, w \Vdash \alpha$, according to the following recursive definition:

(i) $K, w \Vdash p$ iff $\upsilon(w, p) = 1$

(ii) $K, w \Vdash \beta \vee \gamma$ iff $K, w \Vdash \beta$ or $K, w \Vdash \gamma$

(iii) $K, w \Vdash \neg\beta$ iff $K, w \nVdash \beta$

(iv) $K, w \Vdash \Box\beta$ iff for every $t \in W$, $K, t \Vdash \beta$

Definition 4.1.4 *If K is a Kripke structure, then K is a model of* Γ *if for every γ there is a world $w \in W$ such that $K, w \Vdash \gamma$.*

Theorem 4.1.10

(1) Γ *has a model iff it is non-trivial.*

(2) *There are inconsistent sets of formulas that have models.*

Proof: Immediate application of Kripke semantics for $\mathcal{J}$.▌

These results reinforce that $\mathcal{J}$ is a paraconsistent logic.

We can define first-order discussive calculi with necessary identity as follows (these calculi can also be extended to higher-order systems). Let us call $\mathcal{J}^*$ the calculus whose language is that of the quantificational system S5Q$^=$ of Hughes and Creswell (see [148]). We then define:

$$\Gamma \models_{\mathcal{J}^*} \alpha \text{ iff } \Diamond\Gamma \models_{S5Q^=} \Diamond\alpha.$$

In particular, $\models_{\mathcal{J}} \alpha$ iff $\models_{S5Q^=} \Diamond\alpha$. Given these definitions, all the results above can be applied to $\mathcal{J}^*$.

It is possible to present a sound and complete axiomatization for $\mathcal{J}^*$ as follows:

(J1*) If α is an axiom of S5Q$^=$, then $\Box\alpha$.

(J2*) Rules similar to (J2) to (J5) of $\mathcal{J}$ (see page 53).

(J3*) $\Box(\alpha \rightarrow \beta(x)) / \Box(\alpha \rightarrow \forall x \beta(x))$, where x is not free in α.

It is easy to see that in $\mathcal{J}^*$, $\rightarrow_d$, $\wedge_d$, $\vee_d$, $\leftrightarrow_d$, ∇, $\forall$ and $\exists$ have all the classical properties of $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$, $\neg$, $\forall$ and $\exists$, respectively. Furthermore, it is not difficult to justify that: (i) S5Q$^=$ is contained in $\mathcal{J}^*$; (ii) both $\mathcal{J}^*$ and S5Q$^=$ are not decidable; (iii) both have algebraic semantics in which they are sound and complete; (iv) there are inconsistent, but non-trivial sets of formulas of $\mathcal{J}^*$, and finally, (v) a set of formulas Γ is non-trivial iff there is a Kripke structure for $\mathcal{J}^*$ that is a model of Γ. As a result, there are inconsistent sets of formulas that do have models.

There is another, and equivalent, way of building a first-order discussive calculus. This is suitable for certain applications, as will become clear below. We will call this calculus $\mathcal{J}^{**}$ (it is essentially the same as $\mathcal{J}^{*}$). The language of $\mathcal{J}^{**}$ is also the same as that of S5Q$^{=}$, except that if α is a formula, we write $\uplus\alpha$ to denote the formula obtained by preceding α by a sequence of universal quantifiers so that all variables of $\uplus\alpha$ are bound.

To define the calculus $\mathcal{J}^{**}$, we stipulate that

$$\vdash_{\mathcal{J}^{**}} \alpha \text{ iff } \vdash_{S5Q^{=}} \Diamond \uplus \alpha.$$

Furthermore, we say that α is a syntactic consequence of Γ in $\mathcal{J}^{**}$, that is, $\vdash_{\mathcal{J}^{**}} \alpha$, iff there are $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\Diamond\uplus(\gamma_1 \wedge \ldots \wedge \gamma_n \to \alpha)$ is valid in S5Q$^{=}$. The postulates of $\mathcal{J}^{**}$ are (with the usual restrictions):

(J1**) If α is an instance of a classical tautology, then $\Box \uplus \alpha$ is an axiom.

(J2**) $\Box \uplus \alpha, \Box \uplus (\alpha \to \beta) / \Box \uplus \beta$

(J3**) $\Box \uplus (\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta))$

(J4**) $\Box \uplus (\Box\alpha \to \alpha)$

(J5**) $\Box \uplus (\Diamond\alpha \to \Box\Diamond\alpha)$

(J6**) $\Box \uplus (\forall x\alpha(x) \to \alpha(t))$

(J7**) $\Box \uplus \alpha / \alpha$

(J8**) $\Box \uplus \alpha / \Box \uplus \Box\alpha$

(J9**) $\Diamond \uplus \alpha / \alpha$

(J10**) $\Box \uplus (\alpha \to \beta(x)) / \Box \uplus (\alpha \to \forall x\beta(x))$

(J11**) Vacuous quantification may be introduced in any formula.

(J12**) $\Box \uplus (x = x)$

(J13**) $\Box \uplus (x = y \to (\alpha(x) \to \alpha(y))$

Theorem 4.1.11 *In $\mathcal{J}^{**}$, the connectives $\to_d$, $\wedge_d$, $\vee_d$, ∇, $\leftrightarrow_d$, $\forall$ and $\exists$ have all the standard properties of classical (material) implication, conjunction, disjunction, negation, (material) equivalence, and of the universal and existential quantifiers, respectively.*

Thus, classical first-order logic is contained in $\mathcal{J}^{**}$. Furthermore, when we restrict the formulas of $\mathcal{J}^{**}$ to the *stable* ones only – that is, to those formulas α such that $\Box(\alpha \leftrightarrow_d \Diamond\alpha)$ is true – then $\mathcal{J}^{**}$ reduces, in a certain sense, to classical first-order logic.

4.2 Application to the foundational analysis of physical theories

A physical theory T can be characterized roughly as follows. We start with a language L by means of which we express the postulates of T. These postulates, or axioms, can be divided up into three kinds: the logical axioms (say, those of first-order predicate calculus), the mathematical axioms (say, those of a set theory like Zermelo-Fraenkel), and the specific, or physical, axioms of T. It is clear that this schema is quite general, and can be suitably adapted to several contexts (see [97]). In particular, this schema is useful to an analysis of physical theories when we consider the possibility of inconsistencies. For instance, in certain situations, it may be more adequate to use a calculus such as $\mathcal{J}^{**}$ instead of standard first-order logic. This is the case we would like to consider here.

According to Dalla Chiara and Toraldo di Francia (see [117]), the language of a physical theory T is interpreted in a set-theoretic structure of the form

$$\mathcal{A} = \langle M, S, \langle Q_0, \ldots, Q_n \rangle, \rho \rangle,$$

where

(1) M is an instance of a mathematical species of structure in the sense of Suppes-Bourbaki (see [88]).

(2) S is a set of physical situations; that is, a set of physical states assumed by a physical system in a certain interval of time.

(3) Each Q_i, $0 \leq i \leq n$, denotes an operationally defined quantity whose domain of definition is some subset of S. In general, Q_0 represents time.

(4) ρ is a function that associates to each term employed to characterize S, and in particular to each Q_i, a set-theoretic entity in M.

When we measure a physical quantity Q_i of a physical system in a state $s \in S$ in a certain time t_j, we usually consider that its 'acceptable values' $q_i(t_j)$ lie in an interval $[q_i - \epsilon, q_i + \epsilon]$ of the real number line. (The length ϵ depends on the specific measurement technique and on the nature of the quantity in question.) In a certain sense, *all values* in the interval are 'appropriate values' for the measurement of the quantity Q_i of the physical system in a state $s \in S$. For instance, in measuring the table where we are working just now, we should accept (for all 'practical purposes') any value in the interval 1.20 ± 10^{-3} meters.

Let $\alpha(t, q_i(t_j))$ be a formula of L whose only free variables are t and t_j (which stand for time). Dalla Chiara and Toraldo di Francia consider the case of partial formulas, that is, formulas that are not defined for all values of their variables and parameters. But here we will only discuss 'total' formulas. We say that $\alpha(t, q_i(t_j))$ is *true* with respect to a situation s, written $\vDash_s \alpha$, if there are values t^0 of t in the considered time interval and q_i^0 of Q_i in the corresponding interval, such that $\alpha(t^0, q_i^0)$ is true in M in the standard (Tarskian) sense. We also say that $\alpha(t, q_i(t_j))$ is true in $\mathcal{A}$, and write $\mathcal{A} \models \alpha(t, q_i(t_j))$, if $\alpha(t, q_i(t_j))$ is true in M for every $s \in S$. Paraconsistency enters in Dalla Chiara and Toraldo di Francia's approach whenever we get $\overline{t^0}$ and $\overline{q_i^0}$ also in the intervals such that $\neg\alpha(\overline{t^0}, \overline{q_i^0})$ is also true in M. Let us explain this case with an example.

Consider Newton's second law $f = m.a$. The three physical variables appearing in this equation correspond to three deterministic physical quantities: *force* (f), *mass* (m) and *acceleration* (a), which are the physical quantities to be measured. Their acceptable range of values for a certain physical situation s lie, respectively, within three intervals: $[f_1, f_2] \subseteq \mathbb{R}$, $[m_1, m_2], and \subseteq \mathbb{R}$ $[a_1, a_2] \subseteq \mathbb{R}$, each expressing a certain precision ϵ for the measurements. It is the case that $\vDash_s f = m.a$ whenever there are three real numbers $p_1 \in [f_1, f_2]$, $q_1 \in [m_1, m_2]$ and $r_1 \in [a_1, a_2]$, such that $p_1 = q_1.r_1$.

However, due to the imprecision ϵ, there are also other three real numbers p_2, q_2 and r_2, each in the respective interval, so that $p_2 \neq q_2.r_2$, and these numbers are also *acceptable values* for the measurements of the corresponding physical quantities. So, strictly speaking, it is also the case that $\vDash_s \neg(F = m.a)$. In other words, the negation of Newton's law should also be true in the same physical situation s. Hence, we may have, for a sentence α and physical situation $s \in E$, both $\vDash_s \alpha$ and $\vDash_s \neg\alpha$. However, it is not the case that $\vDash_s \alpha \wedge \neg\alpha$, for this last case would entail the existence of three real numbers p', q' and r', belonging to the respective intervals, such that $\vDash_s p' = q'.r' \wedge p' \neq q'.r'$, which is impossible (see [117, p. 66]).

This definition of truth reflects a kind of *empirical truth*. And it has interesting consequences, as Dalla Chiara and Toraldo di Francia point out. For example, the logical connectives are not truth-functional, in the sense that the truth of a conjunction is not equivalent to the simultaneous truth of both conjuncts. Note also the 'paraconsistent aspect' of this definition of truth. After all, we can have both $\vDash_s \alpha$ and $\vDash_s \neg\alpha$. In specifying the underlying logic of Dalla Chiara and Toraldo di Francia's approach, we may use the postulates of a Jaśkowski's discussive logic, such as $\mathcal{J}^{**}$, as logical axioms for such physical theories. (Alternatively, we could use a paraclassic logic instead; see section 7.3.)

4.3 Application to Partial Truth

In a paper by Mikenberg, da Costa and Chuaqui (see [187]), the mathematical concept of pragmatic truth (today called 'partial truth') was introduced. An infinitary logical system was presented to formulate this concept, and some applications were made in logic and in algebra. An application of this concept in the foundations of the theory of probability was studied in [79], and some of its extensions to inductive logic and to the philosophy of science were discussed, respectively, in [92] and [94]. This framework was also employed to examine some issues involved in the theory of acceptance (see [95]), as well as in the modeling of 'natural reasoning' (see [96]). For a general discussion, see [97].

The wide range of applications of the notion of partial truth motivates an investigation of the logic of partial truth. In this section, we will show that there are important connections between this logic and Jaśkowski's discussive logic. We will study these connections in the context of certain 'pragmatic structures', which we will present below. As will become clear, pragmatic structures can be considered as worlds of a Kripke structure, and in this setting, the necessity operator corresponds to the notion of 'pragmatic validity', and the possibility operator to the notion of partial truth. Two systems are then put forward to formalize these notions. One of the main results of the present section is that the logic of partial truth is paraconsistent. The philosophical significance of this result, which justifies the application of partial truth to inconsistent settings, is then discussed.

A remark on our terminology is important here. We call the kind of truth defined in this section *partial truth*. Originally, it was called 'pragmatic truth', due to its connections with the pragmatic conception of truth, as developed by philosophers such as James, Dewey and particularly Peirce (see [187], [79], and [97]). However, our work is not exegetical. The only point to emphasize is that our definition was heuristically inspired by some passages of pragmatic thinkers, such as Pierce, when he wrote: 'consider what effects, that might conceivably have practical bearings, we conceive the object of our conceptions to have. Then, our conception of these effects is the whole of our conception of the object' ([201, p. 31]). In our view, the definition of partial, or pragmatic, truth investigated in this section captures, at least in part, the common concept of a theory *saving the appearances*, usually by means of partially fictitious constructions (see [235] and [52]). We now move on to the formulation of partial truth.

Let us suppose that we are interested in studying a certain domain of knowledge Δ in the field of empirical sciences, for instance, particle physics. We are, then, concerned with certain real objects (in particle physics, with some configurations in a Wilson chamber, some spectral lines, etc.). Let us denote the set of these objects by A_1. Among the objects of A_1, there are some relations that interest us, and that we model as partial relations R_i, $i \in I$ (every relation having a fixed arity). The relations R_i are partial relations, that is, each R_i, supposed of arity r_i, is not necessarily defined for all r_i-tuples of elements of A_1. More formally, an n-place partial relation R can be viewed as a triple $\langle R_1, R_2, R_3 \rangle$, where R_1, R_2, and R_3 are mutually disjoint sets, with $R_1 \cup R_2 \cup R_3 = D^n$, and such that R_1 is the set of n-tuples that (we know) belong to R; R_2 the set of n-tuples that (we know) do not belong to R; and finally R_3 of those n-tuples for which it is not defined whether they belong or not to R. (Note that when R_3 is empty, R is a usual n-place relation that can be identified with R_1.)

The reason for using partial relations is that they are supposed to express what we do know, or what we accept as true, about the actual relations among the elements of A_1. Thus, the partial structure $\langle A_1, R_i \rangle_{i \in I}$ encompasses, so to say, what we know, or accept as true, about the actual structure of Δ. However, to systematize our knowledge of Δ, it is convenient to introduce in our structure $\langle A_1, R_i \rangle_{i \in I}$ some *ideal* objects. (In particle physics, quarks would be an example.) The set of these new objects will be denoted by A_2. It is understood that $A_1 \cap A_2 = \emptyset$, and we stipulate that $A = A_1 \cup A_2$. In this way, the modeling of Δ involves new partial relations R_j, $j \in J$, some of which extend the relations R_i, $i \in I$. Furthermore , there are some sentences (closed formulas) of the language L, in which we talk about the structure $\langle A, R_k \rangle_{k \in I \cup J}$ $(I \cap J = \emptyset)$ that we accept as true, or that are true (in the sense of the correspondence theory of truth). This occurs, for instance, with sentences expressing true decidable propositions (a proposition whose truth or falsehood can be *decided*), and with some general sentences that express laws or theories already accepted as true. Let us denote the set of such sentences, dubbed *primary*, by $\mathcal{P}$ (this set may be empty).

Given this informal discussion, we suggest that what we call a *simple pragmatic structure* (sps) be regarded as a set-theoretic structure of the form:

$$\mathfrak{A} = \langle A_1, A_2, R_i, R_j, \mathcal{P} \rangle_{i \in I, j \in J},$$

where the elements in question satisfy the conditions above. Alternatively, we can simply write:

$$\mathfrak{A} = \langle A, R_k, \mathcal{P} \rangle_{k \in K}$$

for a sps, where $A = A_1 \cup A_2$ and the R_k are partial relations defined on A, and $\mathcal{P}$ is a set of sentences of the language L of the same similarity type as that of $\mathfrak{A}$, and which is interpreted in $\mathfrak{A}$. Note that for some k, R_k may be empty.

Let L be a first-order language with identity, but without function symbols. The symbols of L are logical symbols (connectives, individual variables, quantifiers, and the identity symbol), auxiliary symbols (parentheses), a collection of individual constants, and a collection of predicate symbols. To interpret L in a sps $\mathfrak{A}$ is to associate to each individual constant of L an element of A (the universe of $\mathfrak{A}$), and to each n-ary predicate symbol of L a relation R_k, $k \in K$, of the same arity. It is supposed that every predicate of the family R_k, $k \in K$, is associated with a predicate symbol.

Definition 4.3.1 *Let L and $\mathfrak{A} = \langle A, R_k, \mathcal{P} \rangle_{k \in K}$ be, respectively, a language and a sps in which L is interpreted. Let $\mathfrak{B}$ be a total structure, that is, a usual structure whose n-ary relations are defined for all n-tuples of elements of its universe. And suppose that L is also interpreted in $\mathfrak{B}$. Then, $\mathfrak{B}$ is said to be $\mathfrak{A}$-normal if the following conditions are met:*

(1) *The universe of* $\mathfrak{B}$ is A.

(2) *The (total) relations of $\mathfrak{B}$ extend the corresponding partial relations of $\mathfrak{A}$.*

(3) *If c is an individual constant of L, then in both $\mathfrak{A}$ and $\mathfrak{B}$, c is interpreted by the same element.*

(4) *If $\alpha \in \mathcal{P}$, then $\mathfrak{B} \models \alpha$.*

Given a pragmatic structure $\mathfrak{A}$, it may happen that there are no $\mathfrak{A}$-normal structures. It is possible, however, to provide a system of necessary and sufficient conditions for the existence of such structures (see [187]). One condition of this system is the following: For each partial relation R_k in $\mathfrak{A}$, we construct a set M_k of atomic sentences and negations of atomic sentences such that the former correspond to n-tuples that satisfy R_k, and the latter to n-tuples that do not satisfy R_k (such sentences correspond to n-tuples in the 'anti-extension' of R_k). Let M be the set $\bigcup_{k \in K} M_k$. Therefore, a sps $\mathfrak{A}$ admits an $\mathfrak{A}$-normal structure only if the set $M \cup \mathcal{P}$ is consistent.

In what follows, we will always suppose that our sps satisfies the relevant conditions; in other words, given any sps $\mathfrak{A}$, the set of $\mathfrak{A}$-normal structures is not empty.

Definition 4.3.2 *Let L and $\mathfrak{A}$ be, respectively, a language and a sps in which L is interpreted. We say that a sentence α of L is pragmatically true, or partially true in the sps $\mathfrak{A}$, according to $\mathfrak{B}$, if*

(1) *$\mathfrak{B}$ is an $\mathfrak{A}$-normal structure, and*

(2) *$\mathfrak{B} \models \alpha$, that is, α is true in $\mathfrak{B}$ in accordance with the Tarskian definition of truth.*

In other words, we say that α is *pragmatically (or partially) true* in the sps $\mathfrak{A}$ if there exists an $\mathfrak{A}$-normal structure $\mathfrak{B}$ in which α is true in the standard Tarskian sense. If α is not pragmatically (partially) true in the sps $\mathfrak{A}$ according to $\mathfrak{B}$ (α is not pragmatically (partially) true in the sps $\mathfrak{A}$), we say that α is *pragmatically (partially) false* in the sps $\mathfrak{A}$ according to $\mathfrak{B}$ (α is pragmatically (partially) false in the sps $\mathfrak{A}$).

Given a sps $\mathfrak{A}$, it is natural to consider its $\mathfrak{A}$-normal structures as the worlds of a Kripke structure for S5 with quantification. That is, we have a universe and several structures, defined in this universe, in which the language L can be interpreted, and where every world is accessible to every world (see [148]). It is also natural to extend the language L of the sps $\mathfrak{A}$ to a modal language, by adding the modal operator $\Box$ to the primitive symbols of L. The operator $\Box$, which in modal logic represents the notion of necessity, corresponds in the present situation to *pragmatic validity* (in a sps $\mathfrak{A}$). Analogously, the possibility symbol $\Diamond$, definable in terms of $\Box$ and negation, corresponds to *pragmatic truth* (in a sps $\mathfrak{A}$). Thus, we can extend the semantics of L so that the symbols $\Box$ and $\Diamond$ represent the concepts of pragmatic validity and pragmatic truth, respectively. Moreover, since the universes of all 'worlds' belonging to a sps are the same, it is reasonable that identity behaves, in the cases of pragmatic truth and pragmatic validity, as necessary identity.

Among the pragmatically valid formulas – that is, those formulas α such that $\Box\alpha$ is a theorem of S5 with quantification and necessary identity – there are the logically pragmatically true formulas – that is, those formulas α such that $\Box\Diamond\alpha$ or, equivalently, $\Diamond\alpha$ is a theorem of the same system. To simplify the language, from now on, the former class of formulas will be called *strictly pragmatically valid* and the latter will be called *pragmatically valid*. The first class of formulas coincides with the set of theorems of S5 with quantification and necessary identity; the second, with Jaśkowski's logic associated with the same system.

We first present the logical system that systematizes the notion of strict pragmatic validity. It will be denoted by PV. The system has a language L^* whose primitive symbols are those of a standard formalization of the first-order predicate calculus with identity and individual constants, plus the symbol $\Box$. (For simplicity, function symbols are excluded.) The defined symbols are introduced as usual, and the common conventions in the writing of formulas and in the formulation of postulates (axiom schemas and primitive rules of inference) are employed without explicit mention. The postulates of PV are the following:

(1) If α is an instance of a (propositional) tautology, then α is an axiom.

(2) $\alpha, \alpha \rightarrow \beta / \beta$

(3) $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$

(4) $\Box\alpha \rightarrow \alpha$

(5) $\Diamond\alpha \rightarrow \Box\Diamond\alpha$

(6) $\forall x\alpha(x) \rightarrow \alpha(t)$

(7) $\alpha / \Box\alpha$

(8) $\alpha \to \beta(x) / \alpha \to \forall x \beta(x)$

(9) $x = x$

(10) $x = y \to (\alpha(x) \to \alpha(y))$

In the postulates above, the symbols have a clear meaning. In particular, in the axiom schema (6), t is either a variable free for x in $\alpha(x)$ or an individual constant. This system is essentially S5 with quantification and necessary identity.

We define the concept of deduction as in [146]. The basic idea is that one can only use the generalization rule $\alpha / \forall\alpha$ in a step k of a deduction when a subsequence of the deduction up to k is a proof of α. This restriction to the generalization rule is exactly similar to the one adopted with regard to the necessitation rule, $\alpha / \Box\alpha$. As a result, the usual derived rules, such as the deduction theorem, remain valid.

The semantics of *PV* can be easily developed. The basic (strict) semantic concepts of pragmatic truth, pragmatic falsehood, pragmatic validity, pragmatic invalidity, pragmatic semantic consequence, etc. offer no difficulties in being formulated, and maintain the spirit of Definition 4.3.2. (For more details on these points, see [112] and [114].)

We have the following theorem, whose proof can be obtained by the methods of [148] and [132]:

Theorem 4.3.1 *Let* Γ *be a set of formulas of* L^* *and* α *be a formula of the same language. Then,* $\Gamma \vdash \alpha$ *if, and only if,* $\Gamma \models \alpha$.

That is, α is a *syntactic consequence* of Γ in *PV* iff α is a *strict pragmatic semantic consequence* of Γ.

The logic of strict pragmatic validity, which we have just sketched, can be extended to higher-order (modal) languages, for example by adapting some ideas presented in [132, Chap. 3]. Moreover, we can also develop a metatheoretic study of this logic. For instance, we can adopt different modal systems as basic (S4, for instance), distinguish frames from models etc. Instead of pursuing this path here, we will now consider a different system to study the logic of pragmatic validity. This system, as we will see, is constructed in terms of *PV*, which was presented here mainly as an auxiliary construction. We will then show that the logic of pragmatic validity is paraconsistent.

Let us call *PT* (for 'pragmatic truth') a system whose language is the same as that of *PV*. The idea is to construct a system in which $\vdash \alpha$ means that $\Diamond\alpha$ is strictly pragmatically valid. As before, if α is a formula of L^*, which is the language of *PV* and *PT*, we write $\uplus\alpha$ to denote the formula composed by preceding α by a sequence of universal quantifiers, so that all variables of $\uplus\alpha$ are bound.

Clearly, in order that α be pragmatically valid in the intended sense, we must have that $\vdash \alpha$ in *PT* if, and only if, $\vdash \Diamond \uplus \alpha$ in *PV*. So, *PT* is a kind of Jaśkowski's discussive logic associated with *PV*. (Recall that, given a modal system M, the Jaśkowski's logic associated with M is the set of all formulas α such that $\Diamond\alpha$ is a theorem of M.)

PT can be axiomatized as $\mathcal{J}^{**}$ of the previous section (see postulates J1** to J13** of page 56). The definitions of proof and of (formal) theorem are the usual ones. We now show that that postulates do provide an axiomatization for *PT*.

Lemma 4.3.1 *If α is a theorem of our proposed axiomatization for PT, then $\Diamond \uplus \alpha$ is a theorem of PV.*

Proof: By induction on the length of the proof of α in the proposed axiomatization for PT. Let $\alpha_1, \ldots, \alpha_n$, where α_n is α, be a (formal) proof of α in the proposed axiomatization for PT. Then, α_i, $1 \leq i \leq n$, is an axiom or is obtained by the application of one of the rules. If a_i is an axiom, then it has the form $\Box \uplus \beta$, where β is an axiom of PV (observe that PV is S5 with quantification and necessary identity). Therefore, $\Box \uplus \beta$ is a theorem of PV, and so $\Diamond \uplus \Box \uplus \beta$, i.e. $\Diamond \uplus \alpha_i$, is also a theorem of PV. Suppose that α_i is a consequence of two previous formulas by Rule J2**. Then α_i is $\Box \uplus \beta$, obtained from the premises $\Box \uplus \gamma$ and $\Box \uplus (\gamma \rightarrow \beta)$. By the induction hypothesis, $\Diamond \uplus \Box \uplus \gamma$ and $\Diamond \uplus \Box \uplus (\gamma \rightarrow \beta)$ are provable in PV. Consequently, $\Box \uplus \gamma$ and $\Box \uplus (\gamma \rightarrow \beta)$ are also provable in PV, and so is $\Box \uplus \beta$. But if $\Box \uplus \beta$ is a theorem of PV, then $\Diamond \uplus \Box \uplus \beta$, i.e. $\Diamond \uplus \alpha_i$, is also a theorem. The other rules are treated similarly.▮

Lemma 4.3.2 *If α is a theorem of PV, then $\Box \uplus \alpha$ is a theorem of the proposed axiomatization for PT.*

Proof: By induction on the length of the proof of α in PV. Let $\alpha_1, \ldots, \alpha_n$, where α_n is α, be a proof of α in PV. If α_i, $1 \leq i \leq n$, is an axiom of PV, then $\Box \uplus \alpha$ a is a theorem of the proposed axiomatization for PT, as is easy to see. If α_i is obtained by an application of modus ponens (Rule J2**), from γ and $\gamma \rightarrow \alpha_i$, we have, by the induction hypothesis, that $\Box \uplus \gamma$ and $\Box \uplus (\gamma \rightarrow \alpha_i)$ are provable in the proposed axiomatization. Then, by Rule J2**, $\Box \uplus \alpha_i$ is also provable. Rule J8** is treated analogously.▮

Theorem 4.3.2 *Postulates J1**–J13** characterize PT. That is, we have:*

$$\vdash \alpha \text{ in } PT \text{ iff } \vdash \Diamond \uplus \alpha \text{ in } PV.$$

Proof: Let us suppose that α is a theorem of the proposed axiomatization for PT. Then, by Lemma 4.3.1, $\Diamond \uplus \alpha$ is a theorem of PV. Conversely, assume that $\Diamond \uplus \alpha$ is a theorem of PV. So, by Lemma 4.3.2, $\Box \uplus \Diamond \uplus \alpha$ is a theorem of the proposed axiomatization for PT. Therefore, by Rule J7**, $\Diamond \uplus \alpha$ is a theorem of PT, and so, by Rule J9**, α is also a theorem of PT.▮

Definition 4.3.3 *In PT, we say that the formula α is a syntactic consequence of a set of formulas Γ (in symbols, $\Gamma \vdash \alpha$) if there exist $\gamma_1, \ldots, \gamma_n$ in Γ such that*

$$(\Diamond\gamma_1 \wedge \ldots \wedge \Diamond\gamma_n) \rightarrow \Diamond\alpha$$

is a theorem of PT. Or, equivalently,

$$\Diamond \uplus ((\Diamond\gamma_1 \wedge \ldots \wedge \Diamond\gamma_n) \rightarrow \Diamond\alpha)$$

is a theorem of PV. When $n = 0$, the first formula above reduces, by convention, to α (and $\emptyset \vdash \alpha$ means, thus, that $\vdash \alpha$).

Definition 4.3.4 *A pragmatic theory is a set T of sentences (closed formulas of PT) such that if $\gamma_1, \ldots, \gamma_n$ are in T and $\{\gamma_1, \ldots, \gamma_n\} \vdash \alpha$, then α is also in T.*

It follows that if T is a pragmatic theory and α is a (closed) theorem of PT, then $\alpha \in T$. Let $\mathcal{S}$ be the set of all sentences of PT and T be a pragmatic theory. Using a terminology that has already been introduced, we say that T is *trivial* (*overcomplete*) if $T = \mathcal{S}$; otherwise, T is *non-trivial*. Furthermore, T is *inconsistent* if there is at least one sentence α such that $\alpha \in T$ and $\neg\alpha \in T$, where $\neg$ is the symbol of negation of PT; otherwise, T is *consistent*. We can now prove the following result:

Theorem 4.3.3 *There exist pragmatic theories that are inconsistent but non-trivial.*

Proof: Let c and M be, respectively, an individual constant and a monadic predicate symbol of PT. The theory whose (nonlogical) axioms are $M(c)$ and $\neg M(c)$ is inconsistent. But it is nontrivial, because the corresponding theory of PV, whose (nonlogical) axioms are $\Diamond M(c)$ and $\Diamond\neg M(c)$, is consistent. In effect, it is easy to construct a Kripke model for PV in which both $\Diamond M(c)$ and $\Diamond\neg M(c)$ are true. However, in no Kripke model for PV the formula $\Diamond(M(c) \wedge \neg M(c))$ is true.▌

Given the definitions of the discussive connectives $\rightarrow_d$ and $\wedge_d$, introduced in Definition 4.1.2, we can also prove that, in PT, $\rightarrow_d$, $\wedge_d$, $\vee_d$, $\forall$ and $\exists$ satisfy all the schemas and rules of classical positive logic. If we consider a valid primitive schema (or rule) of classical positive logic, and replace in it implication by discussive implication and conjunction by discussive conjunction, we obtain a valid schema (or rule) of PT, as is easily seen.

Theorem 4.3.4 *If T is a pragmatic theory, then $\alpha \in T$ iff there exist $\gamma_1, \ldots, \gamma_n$ in T such that*

$$(\gamma_1 \wedge_d \ldots \wedge_d \gamma_n) \rightarrow_d \Diamond\alpha$$

is a theorem of PT.

It is worth noting that in some applications of the theory just developed, it is sometimes convenient to employ an alternative definition of syntactic consequence. For instance, in certain applications in the foundations of physics, instead of Definition 4.3.3, it is more appropriate to adopt the following alternative:

Definition 4.3.5 *In PT, the sentence α is said to be a proper syntactic consequence of a set of sentences Γ if there are $\gamma_1, \ldots, \gamma_n$ in Γ such that $\Diamond(\gamma_1 \wedge \ldots \wedge \gamma_n)$ and $\Box((\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow \alpha)$ are theorems of PT (or of PT and some extra axioms).*

The philosophical significance of the above formal account can be seem by considering inconsistencies in our belief systems, as explored by da Costa, French and Bueno (see [112]). If we focus on Theorem 4.3.3, it becomes clear that a pragmatic theory can contain contradictory theorems without becoming trivial. This means that PT belongs to the class of paraconsistent logics. In the case of pragmatic (partial) truth, this is not an unreasonable situation: contradictory propositions may, of course, both be pragmatically true (see [97, Chap. 5]). Thus, partial truth can be used to provide the epistemic framework for characterizing inconsistent belief systems.

More precisely, we can formulate a position according to which 'belief that *p*' is not to be understood as 'belief that *p* is true' in the correspondence sense. When it comes to representational structures, such as scientific theories, ‘belief that *p*’ is to be understood as ‘belief that *p* is pragmatically or partially true’. This allows for the accommodation of inconsistency by acknowledging that an inconsistency is not a permanent feature of reality to which theories correspond, but is rather a temporary aspect of these theories that may turn out to be epistemically fruitful. On this account, it is not the ‘logic of science’, in the sense of the underlying logic of deductive and inferential practices, that is paraconsistent, but rather the appropriate ‘logic of truth’. (For further details, see [97].)

The logic of pragmatic truth, as presented above, has also been developed to serve as a ‘logic of scientific acceptance’ (see [97]). The nature of acceptance is a topic that hasn’t received as much attention as it deserves. The accounts that consider the issue can be divided in two extremes: those that identify acceptance and belief, and those that distinguish the two. The former typically regard belief in terms of the correspondence conception of truth, whereas the latter fall prey to the accusation of some sort of anti-realism (or even conventionalism). An alternative is to retain the connection between belief and acceptance whilst rejecting truth-as-correspondence. On this view, to accept a theory is to be committed, not to believing it to be true *per se*, but to holding it *as if it were true*, for the purposes of further elaboration, development and investigation. Thus, acceptance involves belief that the theory is partially, or pragmatically, true only, and this, we believe, corresponds to the fallibility found in scientific practice.

Linking acceptance and pragmatic truth in this way restores a formal similarity between ‘truth’, taken generally, and acceptance with regard to deductive closure. So, it has been argued, for example, that acceptance differs from truth in that whereas the latter is deductively closed, in the sense that what one deduces from a set of truths is also true, the former generally is not (see [233]). This is correct if closure is understood only in classical terms. However, what the above formal analysis shows is that acceptance, understood within the framework of pragmatic truth, may be regarded as closed under the Jaśkowski’s discussive system. To put it more precisely: although there is no closure with regard to acceptance under classical conjunction and material implication, one can define discussive forms of implication and conjunction as above, with respect to which acceptance can be taken as closed.

This is a result of both general and particular significance. Our contention is that inconsistency can be accommodated in a framework in which accepted propositions are closed under implication. The framework is the one in terms of pragmatic truth, and the form of implication is, of course, discussive. Shifting perspective from the specific to the general, it is the failure to consider such non-classical systems that undercuts the claim that ‘logic’ is not specially relevant to reasoning. Within the framework of pragmatic truth, we can accommodate inconsistency while still retaining a sense of deductive closure. In this manner the relevance of logic to reasoning – especially scientific reasoning – is restored (see [97]).

5 Annotated Logics

Reasoning about inconsistency is also important in computer science, data base theory, and artificial intelligence. For instance, to construct a knowledge base about a certain domain D of knowledge, we generally consult a certain number of experts, say $E_1, \dots, E_n$, of that field of knowledge. Each expert contributes with facts and rules that form the bases $S_1, \dots, S_n$ of sets of sentences. In this way, the whole knowledge base can be taken as the set $S_1 \cup \dots \cup S_n$. However, experts may disagree, and often they do. As a result, this last set may be inconsistent, and so, according to classical first-order model theory, will have no models, and consequently, will be deemed 'meaningless'.

In 1987, H. Blair and V. S. Subrahmanian devised a kind of paraconsistent logic, called 'annotated logic', that was suitable for representing databases and knowledge bases that contain inconsistencies (see [231]). Later, Blair and Subrahamanian developed this framework further, endowing it with a fixed-point theory, a model theory and a proof theory (see [40]). Moreover, they extended earlier results to allow for logic programming over a complete lattice of truth-values, extending accordingly the fixed-point theory and the proof theory (see [39]). In turn, Kifer and Subrahmanian generalized annotated logic in such a way that a framework for logic programming based on the concept of 'bilattice' (which had been developed earlier by Fitting) was provided by the extended annotated logic framework (see [156]). Kifer and Li then showed how annotated logic can be used as a foundation for reasoning in the presence of inconsistency (see [154]). And Kifer and Lozinskii demonstrated an embedding of classical logic in annotated logic; showed the connections between these logics and non-monotonic logics, and devised a mechanical proof procedure for annotated logic (see [155]). Finally, Kifer and Wu showed how annotated logics serve as a foundation for object-oriented databases (see [157]), whereas Kifer and Krishnaprasad showed how annotated logics can be used as a foundation for inheritance networks (see [153]).

The foundational study of annotated logic was suggested by da Costa, Subrahmanian and Vago (see [116]), who developed a family of propositional calculi, called $\mathsf{P}\mathcal{T}$, as well as their first-order counterparts, $\mathsf{Q}\mathcal{T}$. This work has been extended to annotated set theory, and further results about $\mathsf{Q}\mathcal{T}$ have been presented as well (see [108]).

5.1 The annotated logic $\mathsf{Q}\mathcal{T}$

$\mathsf{Q}\mathcal{T}$ is a first-order logic defined as follows. $\mathcal{T}$ is an arbitrary, but fixed, complete lattice. The least element of $\mathcal{T}$ is denoted by $\perp$, while the greatest element is denoted by $\top$. Furthermore, $\neg$ is taken to be a unary operator from $\mathcal{T}$ to $\mathcal{T}$. The language L of $\mathsf{Q}\mathcal{T}$ is a first-order language without identity whose primitive symbols are the following:

1. Connectives: $\rightarrow$ (implication), $\vee$ (disjunction), $\wedge$ (conjunction) and $\neg$ (negation).

2. Individual variables: a denumerably infinite set of variable symbols.

3. Individual constants: an arbitrary family of constant symbols.

4. Quantifiers: $\forall$ (universal quantifier) and $\exists$ (existential quantifier).

5. Function symbols: for each natural number $n > 0$, a collection of function symbols of rank n.

6. Predicate symbols: for each $n \geq 0$, a family of predicate symbols of rank n.

7. Auxiliary symbols: parentheses and comma.

Terms of L are introduced as usual. An (ordinary) *atom* is an expression of the form $P(t_1, \ldots, t_n)$. If P a predicate symbol of rank n and $\lambda \in \mathcal{T}$, an *annotated predicate* is a pair $\langle P, \lambda \rangle$. We will denote an annotated predicate by P_λ, and sometimes will simply call it 'predicate'.

Given an annotated predicate P_λ of rank n and n terms $t_1, \ldots, t_n$, an *annotated atom* is an expression of the form $P_\lambda(t_1, \ldots, t_n)$. The notion of *formula* is introduced in the standard way. Note that the symbol $\neg$ is used here in two distinct ways: first, as a mapping from $\mathcal{T}$ to $\mathcal{T}$, and second, as a unary connective of L. The appropriate meaning of $\neg$ will be given by the context.

Definition 5.1.1 *An interpretation I for the language L is a 4-tuple*

$$I = \langle D, \eta_I, \zeta_I, \chi_I \rangle,$$

where:

(i) D is a non-empty set, called the *domain* of I.

(ii) η_I maps individual constants of L into D.

(iii) ζ_I assigns to each function symbol f of rank n a mapping from D^n to elements of D.

(iv) χ_I assigns to each predicate symbol P of rank n a function $\chi_I(P)$ from D^n to $\mathcal{T}$.

Definition 5.1.2

(i) Let I be an interpretation for L. Then a *variable assignment* v for L, with respect to I, is a map from the set of individual variable symbols of L to D.

(ii) The *denotation* $d_{I,v}(t)$ of a term t of L, with respect to an interpretation I and variable assignment v, is defined inductively as follows:

(a) If t is a constant symbol, then $d_{I,v}(t) = \eta_I(t)$.

(b) If t is an individual variable, then $d_{I,v}(t) = v_I(t)$.

(c) If t is $f(t_1, \ldots, t_n)$, then $d_{I,v}(t) = \chi_I(d_{I,v}(t_1), \ldots, d_{I,v}(t_n))$.

(iii) An annotated atom $P_\lambda(t_1, \ldots, t_n)$ will be denoted by $P(t_1, \ldots, t_n) : \lambda$. An expression of the form $\underbrace{\neg \ldots \neg}_{k\ times}(A : \mu)$, where A is an ordinary atom, is called a *hyper-literal* of order $k, k \geq 0$, and abbreviated by $\neg^k(A : \mu)$.

Definition 5.1.3 *Let I be an interpretation for L, and let v be a variable assignment for L. Also, let A be an ordinary atom, and let α, β, and γ be arbitrary formulas. In this case:*

(i) If A is an ordinary atom $P(t_1, \ldots, t_n)$, then

$$I, v \models (A : \mu) \text{ } \mathit{iff} \text{ } \chi_I(P)(d_{I,v}(t_1), \ldots, d_{I,v}(t_n)) \geq \mu.$$

(ii) $I, v \models \underbrace{\neg \ldots \neg}_{k\ times}(A : \mu)$ *iff* $I, v \models \underbrace{\neg \ldots \neg}_{k-1\ times}(A : \neg\mu)$.

It is important to note here the two senses in which $\neg$ *is used. Inside the atom, that is, in* $(A : \neg\mu)$*, what appears is the map from* $\mathcal{T}$ *to* $\mathcal{T}$*. However, outside the atom, that is, in the first occurrence of* $\neg$ *in the expression* $\neg(A : \neg\mu)$*, it is the negation symbol that should be considered. Condition (ii) states that, to reduce negations, we replace values in the lattice according to the map* $\neg$*.*

(iii) $I, v \models \alpha \wedge \beta$ *iff* $I, v \models \alpha$ *and* $I, v \models \beta$.

(iv) $I, v \models \alpha \vee \beta$ *iff* $I, v \models \alpha$ *or* $I, v \models \beta$.

(v) $I, v \models \alpha \rightarrow \beta$ *iff* $I, v \models \alpha$ *or* $I, v \not\models \beta$.

(vi) If F *is not a hyper-literal, then* $I, v \models \neg F$ *iff* $I, v \not\models F$.

(vii) $I, v \models \forall x\alpha$ *iff for every variable assignment* v' *that agrees with* v *in all variables distinct from* x *(that is,* $v'(y) = v(y)$*, for all* $y \neq x$*),* $I, v' \models \alpha$.

(viii) $I, v \models \exists x\alpha$ *iff for some variable assignment* v' *that agrees with* v *in all variables distinct from* x *(that is,* $v'(y) = v(y)$*, for all* $y \neq x$*),* $I, v' \models \alpha$.

(ix) $I \models \alpha$ *iff for all variable assignments* v *associated with* I*,* $I, v \models \alpha$.

Definition 5.1.4

1. *Let* $\Gamma \cup \{\alpha\}$ *be a set of formulas. We write* $\vDash \alpha$*, and say that* α *is valid (in* $Q\mathcal{T}$*) if, for every interpretation* I*,* $I \vDash \alpha$*. We say that* α *is a semantic consequence of* Γ *iff for every interpretation* I *such that, for all* $\beta \in \Gamma$*,* $I \vDash \beta$*, it is the case that* $I \vDash \alpha$*.*

2. *If* α, β *are formulas of* L*, then* $\alpha \leftrightarrow \beta =_{\text{def}} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

3. *If* α *is a formula, then* $\sim \alpha =_{\text{def}} (\alpha \rightarrow (\alpha \rightarrow \alpha)) \wedge \neg(\alpha \rightarrow \alpha)$.
($\sim \alpha$ *is called the strong negation of* α *in* $Q\mathcal{T}$*.)*

4. *A formula is called complex if it is not a hyper-literal.*

We will describe now an axiomatic system, called $\mathcal{A}$, whose underlying language is L. In the postulates below, α, β and γ denote arbitrary formulas; φ and ψ denote complex formulas, and θ is an annotated atom.

The postulates of our system are:

$(\rightarrow_1)$ $\alpha \rightarrow (\beta \rightarrow \alpha)$

$(\rightarrow_2)$ $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma))$

$(\rightarrow_3)$ $\alpha,\ \alpha \rightarrow \beta / \beta$

$(\rightarrow_4)$ $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$

$(\wedge_1)$ $\alpha \wedge \beta \rightarrow \alpha$

($\wedge_2$) $\alpha \wedge \beta \rightarrow \beta$

($\wedge_3$) $\alpha \rightarrow (\beta \rightarrow \alpha \wedge \beta)$

($\vee_1$) $\alpha \rightarrow (\alpha \vee \beta)$

($\vee_2$) $\beta \rightarrow (\alpha \vee \beta)$

($\vee_3$) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma))$

($\neg_1$) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)$

($\neg_2$) $\varphi \rightarrow (\neg\varphi \rightarrow \alpha)$

($\neg_3$) $\varphi \vee \neg\varphi$

($\exists_1$) $\alpha(t) \rightarrow \exists x \alpha(x)$

($\exists_2$) $\frac{\alpha(x) \rightarrow \beta}{\exists x \alpha(x) \rightarrow \beta}$

($\forall_1$) $\forall x \alpha(x) \rightarrow \alpha(t)$

($\forall_2$) $\frac{\alpha \rightarrow \beta(x)}{\alpha \rightarrow \forall x \beta(x)}$

(τ_1) $(\theta : \perp) \wedge \neg^k(\theta : \mu) \leftrightarrow \neg^{k-1}(\theta : \neg\mu)$

(τ_2) $(\theta : \neg\mu) \rightarrow (\theta : \neg\lambda)$, where $\lambda \leq \mu$.

(τ_3) If $\alpha \rightarrow (\theta : \mu_j)$ for every $j \in J$, then $\alpha \rightarrow (\theta : \mu)$, where $\mu = \sup\{\mu_j : j \in J\}$.

The postulates ($\exists_1$)-($\forall_2$) satisfy the usual restrictions. If $\mathcal{T}$ is a complete lattice, the supremum in rule (τ_3) is well defined. When $\mathcal{T}$ is finite, rule (τ_3) can be replaced by the schema $(\theta : \mu_1) \wedge \ldots \wedge (\theta : \mu_n) \rightarrow (\theta : \mu)$, where $\mu = \sup\{\mu_j : 1 \leq j \leq n\}$.

We easily define the syntactic concepts associated to the axioms above. In particular, the concept of syntactic consequence $\vdash$ is defined as usual. However, the notion of deduction is not finitary if $\mathcal{T}$ is infinite. The propositional counterpart of Q$\mathcal{T}$ can also de developed, as shown in [116].

Theorem 5.1.1 (Soundness) *Let* $\Gamma \cup \{A\}$ *be a set of formulas of* $Q\mathcal{T}$*. Then* $\Gamma \vdash \alpha$ *implies that* $\Gamma \models \alpha$*. That is, the axiomatic system is sound with respect to the semantics of* $Q\mathcal{T}$*.*

Proof: By induction on the length of deduction (if $\mathcal{T}$ is infinite, we use transfinite recursion).▮

Definition 5.1.5 *Suppose that* Γ *is a set of formulas such that the set of annotated constants occurring in* Γ *is finite* (Γ *itself may be infinite). In this case,* Γ *is said to have the finite annotation property.*

We note that if $\mathcal{T}'$ is a substructure of $\mathcal{T}$, then $\mathcal{T}'$ is closed under the operations of $\mathcal{T}$.

Theorem 5.1.2 (Finitary Completeness) *Let* $\Gamma \cup \{A\}$ *be a set of formulas of* $Q\mathcal{T}$*. Then if* $\mathcal{T}$ *is finite or if* $\Gamma \cup \{\alpha\}$ *has the finite annotated property, then* $\Gamma \models \alpha$ *entails* $\Gamma \vdash \alpha$*.*

Proof: By extending the proof of the propositional fragment of $\mathsf{Q}\mathcal{T}$ presented in [116].▮

When $\mathcal{T}$ is infinite, it seems that the completeness can be obtained by adding to the axioms an extra infinitary rule (see [108] for further indications). The system $\mathsf{Q}\mathcal{T}$ is a non-classical logic which is both paraconsistent and paracomplete (see [82]).

5.1.1 Another axiomatization of $\mathsf{Q}\mathcal{T}$.

A different axiomatization of $\mathsf{Q}\mathcal{T}$ can be obtained by adding to the language of classical first-order predicate calculus a symbol for the paraconsistent weak negation satisfying suitable axioms. As an example, let C be an axiomatic systematization of first-order predicate calculus without identity whose symbol of negation is $\sim$. The remaining primitive symbols of C are as those of $\mathsf{Q}\mathcal{T}$. We still suppose that the atomic formulas of C are annotated atoms, as above. Furthermore, our language has a primitive symbol $\neg$ for the weak negation.

Let us denote by $\mathcal{A}'$ the axiomatic system obtained from C by adding to it the axioms $(\neg_1)$, $(\neg_2)$, $(\neg_3)$, (τ_1), (τ_2) and (τ_3) plus the following rule:

[Rule] Let α and β be formulas such that β is obtained from α by replacing sub-formulas of the form $\sim \alpha$ by $\alpha \to ((\alpha \to \alpha) \wedge \neg(\alpha \to \alpha))$, or by replacing sub-formulas of the latter form by the first. We can then infer that $\alpha \leftrightarrow \beta$.

Theorem 5.1.3 *The axiom system* $\mathcal{A}$ *and* $\mathcal{A}'$ *are equivalent. So, both characterize* $Q\mathcal{T}$*.*

Proof: Any postulate of $\mathcal{A}$ is a postulate of $\mathcal{A}'$, and item 3 of Definition 5.1.4 corresponds to a rule in $\mathcal{A}'$. Conversely, any postulate of $\mathcal{A}'$ is a postulate or a definition of $\mathcal{A}$, or is provable in $\mathcal{A}$, as is easy to show.▮

Let X be a non-empty set. A *normal structure based on* X is a function $f : X \times X \mapsto \mathcal{T}$. Let us denote by $\mathsf{Q}\mathcal{T}^2$ the logic obtained from $\mathsf{Q}\mathcal{T}$ by suppressing all function and predicate symbols from the language, with the exception of one binary predicate symbol, which we represent by $\in$. Then $\mathsf{Q}\mathcal{T}^2$ is a dyadic predicate logic whose atoms are annotated by $\mathcal{T}$. These atoms have the form $\in_\lambda (a, b)$, where a and b are terms, and $\lambda \in \mathcal{T}$. This atom will be denoted by $a \in_\lambda b$.

Theorem 5.1.4 $Q\mathcal{T}^2$ *is sound with respect to the semantics of normal structures. If* $\mathcal{T}$ *is finite, or if we consider only sets of formulas having the finite annotation property, then* $Q\mathcal{T}^2$ *is also complete.*

Proof: Consequence of Theorems 5.1.1 and 5.1.2.▮

5.2 Annotated set theory

Theorem 5.1.4 above shows that normal structures are important to annotated logics, particularly when $\mathsf{Q}\mathcal{T}$ is developed along the lines of the previous section. In this section, we will extend annotated logic to set theory, and we will be dealing with normal structures throughout.

Let ZF be a standard formulation of the Zermelo-Fraenkel set theory. The language of annotated set theory, called AZF, is obtained from the language of ZF by adding two individual constants, $\mathcal{T}$ and $\mathcal{U}$. The following axioms are also added to those of ZF:

(AZF.1) $\mathcal{T}$ is a complete lattice, and we denote by $\leq$ an arbitrary, but fixed, ordering. We use $\perp$ and $\top$ to stand for its least and greatest elements, respectively.

(AZF.2) $\mathcal{T} \subseteq \mathcal{U}$ and $\forall x(x \in \mathcal{U} \rightarrow x \subseteq \mathcal{U})$. That is, $\mathcal{U}$ is transitive.

In most applications, it is usually enough to postulate that $\mathcal{T}$ is a set with a reflexive binary relation with unique least and greatest elements.

Definition 5.2.1

(1) We say that $\mathcal{E}$ is a normal structure, or a normal function based on $\mathcal{U}$, if $\mathcal{E}$ is a mapping from $\mathcal{U} \times \mathcal{U}$ into $\mathcal{T}$. We write $x \in_\lambda y$ instead of $\mathcal{E}(x, y) = \lambda$.

(2) If $x \in \mathcal{U}$, then:

(2.1) $x^{(\lambda,\mathcal{E})} =_{\mathsf{def}} \{y : y \in \mathcal{U} \wedge y \in_\lambda x\}$

(2.2) $x^{[\lambda,\mathcal{E}]} =_{\mathsf{def}} \{y : y \in \mathcal{U} \wedge \exists\mu(\mu \in \mathcal{T} \wedge \mu \leq \lambda \wedge y \in_\mu x)\}$

(2.3) $\mathcal{F}_x =_{\mathsf{def}} \{f : f : x \mapsto \mathcal{T} \wedge \exists\mathcal{E}(\mathcal{E}$ is a normal function $\wedge \forall\lambda\forall y(\lambda \in \mathcal{T} \wedge y \in \mathcal{U}) \rightarrow (f(y) = \lambda \leftrightarrow y \in_\lambda x)\}$.

(3) If $x, y \in \mathcal{U}$ and $\lambda \in \mathcal{T}$, then $x =_{\lambda,\mathcal{E}} y =_{\mathsf{def}} \forall z \in \mathcal{U}(z \in_\lambda x \leftrightarrow z \in_\lambda y)$.

(4) A set $x \neq \emptyset$ is strongly transitive if it is transitive and $\forall y(y \in x \rightarrow \mathcal{P}(y) \in x)$, where $\mathcal{P}(y)$ is the power-set of y.

(5) x is called a universe iff it is strongly transitive and for every function $f : x \mapsto x$, if $y \in x$, then $\bigcup ran(f) \in x$, where $ran(f) =_{\mathsf{def}} \{z : \exists t(t \in y \wedge f(t) = z)\}$.

Theorem 5.2.1 *If x is a universe, then x is a standard model of all axioms of ZF, with the possible exception of the axiom of infinity. If $\omega \in x$, then x is a complete universe (ω is the set of natural numbers).*

Set theoretic constructions are used to deal with normal structures based on $\mathcal{U}$. It seems that the more such constructions are available, the better. In most cases, it is useful to take $\mathcal{U}$ as a universe, that is, as a model of ZF if $\omega \in \mathcal{U}$. In particular, we have the following result:

Theorem 5.2.2 *If $\mathcal{U}$ is an universe, $\lambda \in \mathcal{T}$, and $x \in \mathcal{U}$, then $x^{[\lambda,\mathcal{E}]} \in \mathcal{U}$, $x^{(\lambda,\mathcal{E})} \in \mathcal{U}$, and $\mathcal{F}_x \in \mathcal{U}$. Furthermore, $\{y : F(y) \wedge y \in_\lambda x\} \in \mathcal{U}$, where $F(y)$ is any formula of ZF, and $\forall x\forall y(x, y \in \mathcal{U} \rightarrow (\forall\mathcal{E}\forall\lambda(\mathcal{E}$ is a normal function $\wedge \lambda \in \mathcal{T}) \rightarrow x =_{\mathcal{E},\lambda} y) \leftrightarrow x = y)$.*

Proof: Immediate, since $\mathcal{U}$ is a model of ZF (except for the axiom of infinity).▌

We can introduce a weak negation $\neg$ in AZF without difficulty. For instance, it applies to hyper-literals only, and is such that:

$$\neg^k(x \in_\lambda y) \leftrightarrow \neg^{k-1}(x \in_{\neg\lambda} y).$$

5.2.1 Fuzzy sets

The concept of a fuzzy set can be formulated in AZF. If $\mathcal{U}$ is a set, then a *fuzzy set* of $\mathcal{U}$ is a function $u : \mathcal{U} \mapsto [0, 1]$. Let us denote by $\mathcal{F}_{\mathcal{U}}$ the set of all fuzzy sets of $\mathcal{U}$. We say that two fuzzy sets $u, v \in \mathcal{F}_{\mathcal{U}}$ are *equal* iff for every $x \in \mathcal{U}$, $u(x) = v(x)$. Let us use $\mathbf{1}_u$ and $\mathbf{0}_u$ to denote the fuzzy sets of $\mathcal{U}$ so that for every $x \in \mathcal{U}$, $\mathbf{1}_u(x) = 1$ and $\mathbf{0}_u(x) = 0$. Furthermore, if $u, v \in \mathcal{F}_{\mathcal{U}}$ and $x \in \mathcal{U}$, we stipulate that $(u \sqcup v)(x) =_{\mathsf{def}} \sup\{u(x), v(x)\}$; $(u \sqcap v)(x) =_{\mathsf{def}} \inf\{u(x), v(x)\}$, and $\overline{u} =_{\mathsf{def}} 1 - u(x)$.

It is then easy to prove that $\langle \mathcal{F}_{\mathcal{U}}, \sqcup, \sqcap \rangle$ is a complete lattice with the infinite distributive property, and that $\langle \mathcal{F}_{\mathcal{U}}, \sqcup, \sqcap, \overline{\ } \rangle$ is an algebra that, in general, is not Boolean. A fuzzy set u of $\mathcal{U}$ can be identified with a normal structure $\tilde{u}$ based on the set $\mathcal{U} \cup [0, 1]$, such that $\mathcal{T} = \{\top, \bot\}$ and

$$\tilde{u} = \begin{cases} \top & \text{if } x \in \mathcal{U} \wedge y \in [0,1] \wedge y = u(x), \\ \bot & \text{otherwise.} \end{cases}$$

Hence, AZF, considered as the theory of normal structures, encompasses the theory of fuzzy sets. It is clear that if $\mathcal{U}$ is a universe, the definition of fuzzy set in terms of normal structures can be simplified. In the same way, the theory of flou sets and of L-sets (see [198]) can also be obtained by extending the concept of normal structures. Further developments relating annotated logics and fuzzy logics can be found in [108] and [2].

5.3 Applications

Annotated logics have a wide range of applications. In this section, we will illustrate how annotated logics provide a formalism for reasoning about inconsistent knowledge bases.[15] (In section 7.2.3, we will sketch another use of annotated logics.) Although the technical details cannot be given here in complete detail, the references provide additional information.

Expert systems and knowledge bases about a domain D are usually constructed by programmers who, in general, know little about D. To build an expert system, say, in medicine, we need to consult several experts in the particular field we are interested in (say, cardiology). We ask the experts to provide us with adequate knowledge, based on their previous experience, so that we can construct a knowledge base. It is common to assume that the information provided by the experts can be expressed in suitable form in a certain logic language.

But experts usually disagree. For instance, given the same observable symptoms, doctor d_1 may believe that the patient has a virus infection, doctor d_2 may conclude that the patient has an allergic reaction, while doctor d_3 may say that the patient either has a viral infection or an allergy, but not both. It is clear that if we had used the opinions of these three doctors in our knowledge base, we would be led into an inconsistency. The important point is that often experts disagree, and have conflicting opinions, for very good reasons. So, inconsistencies, such as the one found in this sample, should be regarded as *natural*. In other words, to construct knowledge bases, we need to be sensitive to the fact that inconsistencies may be present, and take them seriously.

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[15] This section is partially based on [104].

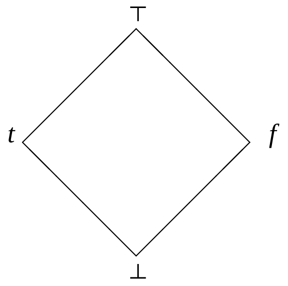

Figure 1: The Lattice FOUR

We present here a general framework for logic programming over a set of truth-values that has the structure of a complete lattice. We develop two formulations of the framework: one that provides a four-valued logic to reason in the presence of inconsistency, and another that shows how to reason with both inconsistency and uncertainty.

Let us assume that we have a fixed set $\mathcal{T}$ of truth-values that is a complete lattice under an ordering $\leq$ defined on $\mathcal{T}$. Let us denote the least upper bound and the greatest upper bound of the subsets $S \subset \mathcal{T}$, respectively, by $\sqcup S$ and $\sqcap S$. For instance, consider the lattice FOUR shown in Figure 1. Here, t and f represent the classical truth-values 'true' and 'false', respectively, while $\perp$ denotes 'unknown', and $\top$ stands for 'inconsistent' or 'over-defined'.

Intuitively, a lattice like FOUR can be useful to accommodate certain classically inconsistent theories. Consider, for example, the theory T axiomatized by $\{p, \neg p, q\}$. T has a model, namely, the interpretation that assigns $\top$ to p and t to q. Given the definition of satisfaction discussed below (see Definition 5.3.3), it can be shown that $\neg q$ is not a logical consequence of T. Of course, this is not the case in classical logic.

Another important complete lattice is SQUARE, that is, the set $SQ = [0,1] \times [0,1]$ of truth-values, where $[0,1] \subset \mathbb{R}$, endowed with the ordering below. Here, the assignment of a truth value $[\mu_1, \mu_2]$ to p means that the degree of belief in p is μ_1, while the degree of disbelief in p is μ_2. The ordering is the following:

$$[\mu_1, \mu_2] \leq [\rho_1, \rho_2] \text{ iff } \mu_1 \leq_{\mathbb{R}} \rho_1 \text{ and } \mu_2 \leq_{\mathbb{R}} \rho_2,$$

where $\leq_{\mathbb{R}}$ is the ordinary 'less than or equal to' defined on the real numbers. In SQ, $[0,0]$ intuitively denotes absolute lack of belief, $[0,1]$ denotes complete disbelief, $[1,0]$ denotes complete belief and $[1,1]$ denotes absolutely inconsistent beliefs.

Let $(A : \mu)$ be an annotated atom over $\mathcal{T}$ (see Definition 5.1.2). If α_1 and α_2 are first-order expressions (terms or atoms), then a *substitution* θ of variable symbols for terms is called a *unifier* of α_1 and α_2 iff the application of θ to α_1, denoted $\alpha_1\theta$, yields the same expression as $\alpha_2\theta$. A *most general unifier* (mgu for short) of any two syntactic expressions α_1 and α_2 is a unifier θ such that for any unifier ϑ of the expressions α_1 and α_2, there is a substitution γ such that $\theta\gamma = \vartheta$. If α_1 and α_2 are unifiable terms of atoms, then they posses a mgu (see [104]).

Definition 5.3.1 *If $L_0, L_1, \ldots, L_n$ are annotated atoms over $\mathcal{T}$, then $L_0 \Leftarrow L_1 \wedge \ldots \wedge L_n$*

is an annotated clause over $\mathcal{T}$. L_0 is called the head of the annotated clause, while $L_1 \wedge \ldots \wedge L_n$ is its body. We will often refer to annotated clauses just as clauses.

Definition 5.3.2 *An annotated logic program (ALP) over $\mathcal{T}$ is a finite set of annotated clauses over $\mathcal{T}$.*

5.3.1 Semantics

Let us define a semantics for ALPs. To do that, we will consider only those interpretations whose domain of discourse are the set of ground terms of the language (Herbrand interpretations). An interpretation I of an ALP P over $\mathcal{T}$ is a mapping $I : Bp \mapsto \mathcal{T}$, where Bp is the Herbrand base of P, that is, the set of variable-free atoms expressible in the language of P. The ordering $\leq$ is extended to the interpretation in a natural way, namely:

$$I_1 \leq I_2 \text{ iff } (\forall A \in Bp)(I_1(A) \leq I_2(A)).$$

The orderings $\geq$, $<$ and $>$ are defined in the usual way. We also assume the existence of a function $\neg : \mathcal{T} \mapsto \mathcal{T}$.

Definition 5.3.3 (Satisfaction) *An interpretation I is said to satisfy*

1. the formula α iff it satisfies every closed instance of α;
2. the variable-free annotated atom $(A : \mu)$ iff $I(A) \geq \mu$;
3. the variable-free annotated hyper-literal $\neg(A : \mu)$ iff $I(A) \geq \neg(u)$;
4. the variable-free formula $\alpha_1 \wedge \alpha_2$ iff I satisfies α_1 and α_2;
5. the variable-free formula $\alpha_1 \vee \alpha_2$ iff I satisfies α_1 or α_2;
6. the variable-free formula $\alpha_1 \Leftarrow \alpha_2$ iff either I satisfies α_1 or does not satisfy α_2;
7. the variable-free formula $\alpha_1 \Leftrightarrow \alpha_2$ (that is, $(\alpha_1 \Leftarrow \alpha_2) \wedge (\alpha_2 \Leftarrow \alpha_1)$) iff I satisfies $\alpha_1 \Leftarrow \alpha_2$ and $\alpha_2 \Leftarrow \alpha_1$;
8. the closed formula $\exists x\alpha$ iff there is some variable-free term t such that I satisfies $\alpha[x/t]$ (the result of replacing all free occurrences of x in α by t);
9. the closed formula $\forall x\alpha$ iff for every variable-free term t, I satisfies $\alpha[x/t]$.

When I satisfies α, we write $I \models \alpha$, and $I \not\models \alpha$, when it does not. In this section, we write $(\forall)\alpha$ and $(\exists)\alpha$ to denote $\forall x_1 \ldots \forall x_n\alpha$ and $\exists x_1 \ldots \exists x_n\alpha$, respectively, where $x_1, \ldots, x_n$ are the free variables of α.

Lemma 5.3.1 *If I is an interpretation, then:*

1. $I \models \neg(A : \mu)$ iff $I \models (A : \neg\mu)$.
2. $I \models (\exists)(\neg(A : \mu))$ iff $I \models (\exists)(\alpha : \neg(\mu))$.

Theorem 5.3.1 *Suppose that P is an ALP over $\mathcal{T}$. Let P' be the ALP obtained from P by replacing all annotated literals of the form $(\neg A : \mu)$ by $(A : \neg(\mu))$. Then I is a model of P iff I is a model of P'.*

Without loss of generality, we will assume (throughout this work) that ALPs contain no negated literals. Moreover, associated with every ALP P over $\mathcal{T}$, there is a function $\mathcal{T}_p$, from the class of Herbrand interpretations to the class of Herbrand interpretations, defined as follows:

$$\mathcal{T}_p(I)(A) =_{\text{def}} \sqcup\{\mu : (A : \mu) = \beta_1 \wedge \ldots \wedge \beta_n \text{ is a ground instance of an annotated clause in } P \text{ and} I \models \beta_1 \wedge \ldots \wedge \beta_n\}.$$

Theorem 5.3.2 *Suppose that P is an ALP over $\mathcal{T}$ (where $\mathcal{T}$ is a complete lattice under $\leq$) and that $\mathcal{T}_p$ is as above. Then I is a model of P iff $\mathcal{T}_p(I) \leq I$.*

Theorem 5.3.3 *Suppose that P is an ALP over $\mathcal{T}$ as above. Then $\mathcal{T}_p$ is monotonic; that is, $I_1 \leq I_2$ entails $\mathcal{T}_p(I_1) \leq \mathcal{T}_p(I_2)$.*

The monotonicity of $\mathcal{T}_p$ guarantees, by the Tarski-Knaster theorem, that $\mathcal{T}_p$ has a least fixed point that coincides with the least pre-fixed point of $\mathcal{T}_p$. (Here, I is a pre-fixed point of $\mathcal{T}_p$ iff $\mathcal{T}(I) \leq I$.)

Theorem 5.3.4 *P has a least model that is identical with the least fixed point of $\mathcal{T}_p$.*

Given that $\mathcal{T}$ is a complete lattice, it possesses a least element and a greatest element, which we denote, respectively, by $\perp$ and $\top$. Furthermore, associated with every $\mathcal{T}$, there are two distinguished interpretations, denoted by $\triangle$ and ∇, that assign the truth-values $\perp$ and $\top$, respectively, to the elements $A \in Bp$, where Bp is the Herbrand base of P.

Definition 5.3.4 *If P is an ALP over $\mathcal{T}$, then the upward iteration of $\mathcal{T}_p$ is defined, for all ordinals λ, as*

$$\begin{cases} \mathcal{T}_p \uparrow 0 =_{\text{def}} \triangle \\ \mathcal{T}_p \uparrow \lambda =_{\text{def}} \sqcup_{\alpha<\lambda}\mathcal{T}_p(\mathcal{T}_p \uparrow \alpha). \end{cases}$$

Theorem 5.3.5 *$\mathcal{T}_p \uparrow \omega$ is identical to the least fixed point of $\mathcal{T}_p$.*

Definition 5.3.5 *A model I of the ALP P over $\mathcal{T}$ is supported iff $I(A) = \sqcup\{\mu : (A : \mu) \Leftarrow (\beta_1 : \mu_1) \wedge \ldots \wedge (\beta_n : \mu_n)$ is a ground instance of an annotated clause in P and $I \models (\beta_1 : \mu_1) \wedge \ldots \wedge (\beta_n : \mu_n)\}$.*

Theorem 5.3.6 *I is a fixed point of $\mathcal{T}_p$ iff I is a supported model of P.*

Definition 5.3.6 *Let λ be an ordinal. The downward iteration of $\mathcal{T}_p$ is defined as follows:*

$$\begin{cases} \mathcal{T}_p \downarrow 0 =_{\text{def}} \nabla \\ \mathcal{T}_p \downarrow \lambda =_{\text{def}} \sqcap_{\alpha<\lambda}\mathcal{T}_p(\mathcal{T}_p \uparrow \alpha) \end{cases}$$

Definition 5.3.7 *The ALP is canonical iff* $\mathcal{T}_p \downarrow \omega$ *is a fixed point of* $\mathcal{T}_p$.

Definition 5.3.8 *Suppose* C_1 *and* C_2 *are the annotated clauses given below:*

$$\begin{cases} (A^1 : \mu^1) \Leftarrow (B^1_1 : \rho^1_1) \wedge \ldots \wedge (B^1_n : \rho^1_n) \\ (A^2 : \mu^2) \Leftarrow (B^2_1 : \psi^2_1) \wedge \ldots \wedge (B^2_m : \psi^2_m). \end{cases}$$

C_1 *is semantically equivalent to* C_2 *iff there is a substitution* Θ *such that* $A^1\Theta = A^2$ *and*

$$\{(B^1_1\Theta : \rho^1_1), \ldots, (B^1_n\Theta : \rho^1_n)\} = \{(B^2_1 : \psi^2_1), \ldots, (B^2_m : \psi^2_m)\}.$$

Definition 5.3.9 *Suppose that* C_1 *and* C_2 *do not share common variables. An ALP* P *is closed iff for every pair* C_1, C_2 *of annotated clauses in* P *satisfying the previous condition, if* C_1, C_2 *are of the form*

$$\begin{cases} (A^1 : \mu^1) \Leftarrow (B^1_1 : \rho^1_1) \wedge \ldots \wedge (B^1_n : \rho^1_n) \\ (A^2 : \mu^2) \Leftarrow (B^2_1 : \psi^2_1) \wedge \ldots \wedge (B^2_m : \psi^2_m), \end{cases}$$

such that A^1, A^2 *are unifiable via the mgu* Θ, *and* μ^1 *and* μ^2 *are incomparable (i.e.,* $\mu^1 \nleq \mu^2$ *and* $\mu^2 \nleq \mu^1$*), it is the case that*

$$((A^1 : \sqcup\{\mu^1, \mu^2\}) \Leftarrow (B^1_1 : \rho^1_1) \wedge \ldots \wedge (B^1_n : \rho^1_n) \wedge (B^2_1 : \psi^2_1) \wedge \ldots \wedge (B^2_m : \psi^2_m))\Theta$$

is semantically equivalent to some annotated clause in P. *The closure of an ALP* P, *denoted* $CL(P)$, *is the closed ALP obtained by repeatedly adding to* P *all clauses* C *obtained from annotated clauses* C_1, C_2 *whose heads are unifiable and whose heads's annotations are incomparable.*

Every ALP P can be extended to a closed ALP $CL(P)$ by adding a finite number of new annotated clauses, as is easy to show. Moreover, we have:

Theorem 5.3.7 *Suppose* P *is an ALP over a complete lattice* $\mathcal{T}$ *of truth-values. Then:*

(1) $\mathcal{T}_p = \mathcal{T}_{CL(P)}$

(2) *Hence,* P *and* $CL(P)$ *have the same models, i.e., they are logically equivalent.*

(3)*If* I *is a supported model of* $CL(P)$, *and* A *is a variable-free atom such that* $I(A) = \lambda \neq \perp$, *then there is a single annotated clause in* P *having a ground instance of the form*

$$(A : \mu) \Leftarrow (B_1 : \psi_1) \wedge \ldots \wedge (B_n : \psi_n)$$

such that $\lambda \leq \mu$ *and* $I \models (B_1 : \psi_1) \wedge \ldots \wedge (B_n : \psi_n)$.

Definition 5.3.10 P *is canonical iff* $\mathcal{T}_p \downarrow \omega$ *is the greatest fixed point of* $\mathcal{T}_p$.

Theorem 5.3.8 *If* P *is canonical, then* $\mathcal{T}_p \downarrow \omega$ *is the greatest supported model of* P.

5.3.2 Executing queries

Important applications of the formal framework above are found when we analyze the interactions between an user who ask queries to a knowledge base. To discuss cases of this type, we assume that all ALPs are closed, which suffices for this purpose (see [104]). In this kind of application, the knowledge base is expressed as an ALP over a suitably chosen complete lattice $(\mathcal{T}, \leq)$ of truth-values. In this framework, experts may use expressions such as *p is likely to be false* or, more precisely, *p is false with 90 % certainty* (see [104], where examples are given).

Other applications of annotated logics will be mentioned in the next section.

6 Developments in paraconsistent logic

6.1 Some carried out developments

Since the 1960s, when paraconsistent logic was established as a logic *stricto sensu*,[16] several developments have been made, most of them in connection with the C-systems.[17]

To give an idea (although not a complete one) of some of these developments,[18] let us recall *some* of the most important events. In 1969, M. Fidel proved the decidability of C_n, $1 \leq n \leq \omega$, by algebraic methods (see [128], and also [124]). Another decision method for C_n, $1 \leq n \leq \omega$, was presented by D. Marconi in 1980 using semantic tableaux (see [180]). In 1987, W. Carnielli, after systematizing finite many-valued logics through tableaux, approached C_1 by this method, and also showed that C_1 is decidable (see [57]).

In 1970s, da Costa, Alves, Loparić and Arruda studied the semantic counterpart of the calculi C_n, $1 \leq n \leq \omega$, later extended to the calculi $C_n{}^{\star}$ and $\mathcal{D}_n$, $1 \leq n \leq \omega$ (see [20]). Da Costa and Alves' work on the issue has been mentioned already. Although the subject has not been discussed here, new hierarchies of calculi, constructed between C_n and C_{n+1}, $n \geq 0$, have also been introduced. Their semantics was studied by Alves (see [9]). We noted above that Alves has proved the decidability of da Costa's propositional systems by the method of quasi-matrices. Later, Loparić presented a two-valued semantics and a decision method for C_ω (see [171], [172], and [173]).

Arruda and da Costa have also axiomatized some paraconsistent systems that are also relevant logics (see [21]). In their systems, $\alpha \wedge (\alpha \to \beta) \to \beta$, the rules of contraction, namely, $\alpha \to (\alpha \to \beta)/\alpha \to \beta$ and $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$, and the deduction theorem do not hold. These systems (called P and $P^{\star}$), as well as their quantificational counterparts, are not finitely trivializable, are not decidable by finite matrices, but can be extended to modal and tense logics, as these authors have shown. Routley and Loparić studied the semantic aspects of P, as well as some of its 'dialectical' extensions (see [221]). Arruda and da Costa have also developed the systems J_n, $1 \leq n \leq 5$ (at the propositional and predicate levels), in which modus

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[16] That is, with the development of (at least) first-order predicate calculus. This turning point is acknowledged in general (see, for instance, [232]).

[17] Further details on these developments and references to the authors who have contributed to the subject can be found in [15], [16], [102], and [124].

[18] For further information, see the references just mentioned.

ponens is not valid (see [19]). These systems have been studied also by M. Bunder (see [54]).

Based on Jaśkowski's ideas, D'Ottaviano and da Costa introduced a three-valued propositional logic J_3 with two designated values, which turn out to be paraconsistent (see [125]). Furthermore, D'Ottaviano presented an axiomatization of J_3 (see [121] and [122]), and studied the connections between this calculus and several others, such as intuitionistic logic and Łukasiewicz's three-valued logic. The system J_3 was further studied by D'Ottaviano herself in other papers, where she introduced the concept of J_3-theories, and adapted and proved to the latter several results of model theory (see also [123]).

The algebraization of Jaśkowski's logic was studied by Kotas, following the axiomatic treatment given by da Costa, Dubikajtis and Kotas himself. In particular, Kotas proved that the system is not decidable by finite matrices (see [162]). Other studies related to Jaśkowski's logics have been developed in [131], [163], [164], [165], and [179]. Pinter's system that deals with 'inherent ambiguity' is a slight modification of a Jaśkowski logic (see [208]).

One of the main problems with regard to the algebraic study of the C-logics is that the only congruence relation in C_1 is the identity relation, as shown by C. Mortensen (see [190]). Despite this fact, some algebraic approaches to the C-systems, or their extensions, were presented in [71], [72], [103], [227], [58], and [170], as well as in some of Béziau's works referred to below.

The fact that the C-systems do not enable substitutivity by equivalents is discussed by I. Urbas (see [234]), who attempted to remedy this situation by extending these systems with the addition of new rules. He then proved that these extensions do not lead to systems distinct from classical logic, concluding that new hierarchies should be constructed, where adequate equivalence relations can be formulated. The same problem regarding the impossibility of defining such equivalences in da Costa's systems is discussed by Peña in [203, pp. 284ff].

Important contributions to PL have been made by J. -Y. Béziau in a series of works. He started studying PL in the late 1980s, formulating a semantics for C_1. He also axiomatized a sequent system for C_1. (This was a problem that A. Raggio tried to solve in 1960s; see [124].) Béziau also extended C_1 to a stronger system, C_1+, by replacing the axioms $\alpha^o \wedge \beta^o \to (\alpha \odot \beta)^o$, where $\odot \in \{\wedge, \vee, \to\}$, by $\alpha^o \vee \beta^o \to (\alpha \odot \beta)^o$. As a result, he obtained more theorems as well as some of De Morgan laws. He then studied a non-truth functional semantics for the stronger system (see [31]). Béziau also investigated a general theory of negation (see [32]), and argued that both classical first-order logic and the modal system S5 can be viewed as paraconsistent systems (see [35]). These last results led him to study a new theory of opposition, where a polyhedron replaces the traditional square of opposition (see [37]).

The extension of first-order paraconsistent logic to set theory has already been discussed in section 3. This work was initiated by da Costa (see [67]), and continued by Arruda (see [13]). (For additional references and historical details, see [124] and [110].) Higher-order paraconsistent logics corresponding to the C-systems have been presented in [11], [83], and [84]. Semantics for paraconsistent systems containing descriptions and Hilbert's ϵ-symbol have been discussed, respectively, by Abar (see [1]) and Yamashita (see [237]). These authors have also shown how to develop a general theory of v.b.t.o. in the calculi $C_n^=$.

These developments are, of course, only part of what has been done in paraconsistent logic. Despite the huge amount of work that has been produced in the area (which we cannot, of course, even begin to summarize here), we still do not have a comprehensive introductory book on the subject. (A first introductory book on PL is Grana's [134]; see also [135; 136; 137; 138] for some of his ideas on the subject.) We will now briefly comment on some other lines of research.

6.2 A taxonomy of C-systems

Recently, the C-logics have been studied from a different perspective. In 'A Taxonomy of C-systems', Carnielli and Marcos presented an elaborate study of the foundations of paraconsistent logics (see [59]). In particular, the authors show how several classes of paraconsistent logics can be distinguished from the point of view of general abstract logic. They also put forward a new discriminating account of several logical principles, such as the principle of non-contradiction and various forms of pseudo-Scotus (also known as the principle of explosion).

The logics of formal inconsistency (LFIs) are then introduced as a large class of paraconsistent logics in which the concepts of consistency and inconsistency can be internalized. This allows Carnielli and Marcos to present a novel account of the notion of consistency, and distinguish formally between the notions of contradictoriness and inconsistency. While studying the general features of an important subclass of LFIs – namely, the C-systems and the dC-systems – they show that most paraconsistent logics can be seen as C-systems (C-logics), and explore their properties and shortcomings.

Developing further the work done in [59], Carnielli, Coniglio and Marcos improved some results and emphasized semantic and proof-theoretic aspects of the logics of formal inconsistency (see [60]). The main LFI and one of its primary subclasses, the C-systems, were also surveyed. A striking feature of LFIs is their ability to encode classical logic, in the sense that LFIs are able to reproduce classical reasoning, despite being subsystems of classical logic. The dC-systems, a particular subclass of the C-systems, are carefully discussed; particular cases include da Costa's C_n, and Jaśkowski's D2.

By adding new axioms to the dC-systems, the authors show that it is possible to introduce a large family of logics by controlling the propagation of inconsistency. Exploring this possibility allows the definition of thousands of new logics. The paper also emphasizes the semantic meaning of LFIs, discussing in detail both the valuation semantics for LFIs and the possible-translation semantics. Another kind of semantics, the society semantics, is also addressed. Furthermore, modal extensions of LFIs and their Kripke semantics are discussed, as well as some first-order LFIs. An entire subsection of [60] is dedicated to proof systems, especially to the tableaux method; indeed, tableaux proof systems for several LFIs are provided. Several other issues, including the difficulties of algebraizing LFIs, are also studied in both papers.

A further development of this approach is concerned with modal LFIs. Given the epistemic version of the normal logic KT, if we add to it the so-called verificationist thesis, 'A implies that it is possible to know A', it follows that 'A implies that KA'. This yields the collapse of the knowledge operator. This problem is known as the knowability paradox, and it threatens any normal modal logic that expands KT by the alethic thesis. Modal LFIs are useful in the task of avoiding the knowability paradox, because it is possible to add to the paraconsistent version of the modal logic KT the verificationist thesis without losing the modality, as shown in [61]. In this sense, modal LFIs are able to avoid the knowability paradox while simultaneously recovering classical reasoning with such logics.

6.3 Other directions

Several directions, other than those suggested by the development of da Costa's C-logics, were also proposed to deal with inconsistencies. For instance, D. Batens initially studied paraconsistent systems related to dialectics (see [28]). Later, he considered the case of inconsistency-adaptive logics, i.e. logics in which abnormalities are inconsistencies, and presented two first-order inconsistency-adaptive logics, both from the proof-theoretic and the model-theoretic points of view (see [29] and [30]).

In 1979, Lorenzo Peña introduced a new kind of PL that resembled fuzzy logic in certain respects (see [202]; see also [203; 204], where Peña's ideas are developed further).

Paraconsistent logic has also progressed in Australia and New Zealand, in part due to the effords of R. Routley, motivated by the connections with relevant logics. In these logics, certain schemas and rules from classical logic, such as $\alpha \to (\neg\alpha \to \beta)$ and $\alpha, \neg\alpha \vdash \beta$, are not valid (see [12], [224], and [222]). As a result, these systems can also be used as the basis for inconsistent but non-trivial theories. Marconi gave a precise formulation of the relation between paraconsistent logic and relevant logic (see [179]). From our point of view, most relevant systems are paraconsistent. Additional contributions from Australia can be found in the works of Mortensen [190], Bunder [53; 54], and Priest [211; 213; 212] (see also [214] for Priest's view on paraconsistent logics).

Another domain to which paraconsistent logic can be applied is dialectics. From a historical perspective, dialectics may implicitly assume some system of paraconsistent logic, since most dialectical views maintain that there are 'real contradictions', i.e., that the actual world is contradictory. (For a basic antology on this subject, see [179].)

We mention here the researches of da Costa and R. G. Wolf on the underlying logic of dialectics – dialectics conceived according to the interpretation of McGill and Parry (see [185]). The resulting systems, involving propositional and predicate levels, have formal features analogous to the C-logics. But they are strictly stronger than the latter (for details, see [106] and [107]).

In [5], S. Akama reappraises D. Nelson's work on inconsistent systems from the 1950s (see [199; 200]). Taking Nelson as a forerunner of PL, Akama presents an interesting study on Nelson's 'constructive' PL.[19] Additional developments of PL, more closely related to applications, will be discussed in the next section.

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[19] Akama claims that 'da Costa did not appear to be familiar with Nelson's system' (see [5]). However, Nelson's paper [200] is listed in the references of da Costa's seminal work [67].

7 Applications

7.1 Technology

As we discussed above, annotated logics were originally introduced by Blair and Subrahmanian in the 1980s. These logics have been developed further and applied to several fields, such as robot control [195], air traffic control [196], control systems for autonomous machines, defeasible deontic reasoning [194], information systems [6] and medicine. Here, we cannot do justice to all applications that have been devised in recent years. But let us give a brief sketch of some of the recent developments in the area.

A programming language, called PARALOG, was implemented in [8]. It is a paraconsistent version of PROLOG, and it has been used to construct several computational systems for planing data, vision systems, and to represent inconsistencies (see [7]).

Furthermore, digital circuits, inspired in annotated logics, were introduced in [3] to accommodate incompatible signals. The authors suggest that this device may be useful to develop more general electric circuits, as well as to articulate applications in logistic and to decision procedures. Hardware devices are also under construction, motivated by paraconsistent ideas. The so-called *para-analyser* enables scientists to handle uncertainty, inconsistencies and paracompleteness (see [4]). Several other related devices have also been constructed, leading to the first ‘paraconsistent robot’: Emmy (see [229] and [230]).

Interesting applications are being developed in medicine: in the recognition of cancer cells, in Alzheimer illness, and in disfunctions of speech. These applications are new, and they suggest that applied paraconsistent logics may have an extraordinary role. Furthermore, in engineering, non-monotonic and defeasible forms of reasoning have been represented in terms of paraconsistent logics, leading to the development of softwares that are being used in traffic control – of trains, aircrafts, and cars (see [192] and [193]). The hardware counterpart has also been developed in the form of a computer chip (see [197]).

7.2 Informatics

Several types of applications of paraconsistent logics in informatics have been devised. In this section, we will discuss some of them.

7.2.1 Epistemic inconsistencies in artificial intelligence

Paraconsistent logic has been used in the construction of models of ‘real life reasoning’ in artificial intelligence. This is part of the general problem of ‘practical reasoning’, and is treated under the label of the ‘logic of appearance’. In all of these areas, we find contributions by T. Pequeno, A. Buchsbaum, A. T. Martins, and their collaborators (see, e.g., [51], [184], [207], [62], and [205]). For example, they introduce the notion of epistemic inconsistency, which refers to the existence of contradictory views about the same situation (see [206]). These contradictions reflect the incompleteness (or vagueness) of our knowledge about the domain under study. In particular, the association of this phenomenon with non-monotonic reasoning is explored. The idea is to devise a logical system and the corresponding semantics that make these notions precise and enable reasoning on these inconsistent views without triviality (see [205]).

In [49], it is presented a proof method for automation of reasoning in paraconsistent logic, using da Costa's calculus $C_1^\star$. The method is analytical, and it employs a specially designed tableaux system. In fact, the authors present two tableaux systems. The first, with a small number of rules, is used to prove the soundness and the completeness of the method. The second, which is equivalent to the first, is a system of derived rules designed to enhance computational efficiency. A prototype based on the second system was also effectively implemented.

7.2.2 Other paraconsistent fuzzy systems

In addition to the systems connecting paraconsistent logic and fuzzy logic mentioned earlier, Barreto and Ebecken presented some applications of PL in artificial intelligence. They explored, in particular, the use of PL in the construction of paraconsistent knowledge bases, implemented in fuzzy shells (see [24], [25] and [26]). It was shown that it is possible to build paraconsistent knowledge bases in the Mathlab Fuzzy Logic Toolbox, and that fuzzy shells are inconsistency-tolerant. Moreover, it has also been shown that the 'defuzzification method', that plays an important role in paraconsistent fuzzy systems, must be chosen according to the nature of the knowledge base in question (see [27]). In fact, the authors describe a particular defuzzification method in the interpretation of a paraconsistent knowledge base.

These works provide only partial results for the handling of inconsistency in artificial intelligence (AI), and the authors insist on the need for investigating the problems of coherence and normalization of inconsistent and paraconsistent knowledge bases, including their semantic aspects. In [23], the relationship between paraconsistent knowledge bases and possibilistic logic is explored. Using possibilistic logic, it is possible to accommodate inconsistent information by assigning to each piece of knowledge a certain degree of paraconsistency, and by tracking the use of such pieces in the derivation of new conclusions. This proposal seems to solve partially the problem of normalization and the problem of knowing whether paraconsistent propositions in the knowledge base affect the conclusions of the deductive processes where they are used. The degree of paraconsistency and the tracking system help to indicate to what extent the conclusions are thus affected.

Furthermore, in [24], the proof of the consistency between possibilistic resolution and Zadeh's approximate reasoning theory is outlined. This is an interesting result for AI, since it provides a theoretical base for the improvement of an efficient methodology for reasoning in the presence of inconsistency. An interesting aspect of this methodology is that it provides a possible model of human knowledge, where qualitative analysis is needed. Examples of applications include medical diagnosis as well as juridical and business decisions.

7.2.3 The matrix connection method and paraconsistency

As is well known, W. Bibel's matrix connection method provides an alternative procedure for theorem proving other than the usual resolution technique (see [44] and [45]). This method was adapted and implemented in the particular case of an annotated propositional paraconsistent logic by Kaestner, Krause, Musicante and Nobre (see [152] and [166]). We will give here the main ideas of this approach, and highlight some connections between annotated logic and computation.

We will be working, once again, in a propositional language with standard connectives and other symbols. Let A be a non-empty, finite, ordered set of propositional symbols. To each element $a \in A$, we associate a non-empty, finite lattice $\mathcal{T}_a$. The elements of $\mathcal{T}_a$ are called *annotated constants*, and denoted by μ, ν etc.

Definition 7.2.1 *A* ground literal *is a triple* (a, μ, p), *where* $a \in A$, $\mu \in \mathcal{T}_a$, *and* $p \in \{0, 1\}$. p *is the* polarity *of the literal. We will write either* $\sim L$ *or* $\sim L_\mu$ *to denote the literal* $(a, \mu, 1)$, *while either* L *or* L_μ *denotes the literal* $(a, \mu, 0)$. *In general, literals will be denoted by* K, L, M *etc.*

Let R be an alphabet of *occurrences* or *positions*. The elements of R are denoted by r.

Definition 7.2.2 *By induction, we define the concept of* propositional matrices *over* (A, R), *denoted by the letters* D, E, F. *Similarly, we define their* size $\sigma(F)$, *their* positions $\Omega(F) \subset R$, *and the* depth $\delta(r)$ *of* r *in the matrix* F, *for any* $r \in \Omega(F)$*:*

a) *For any literal* L, *and for any* $r \in R$, *the pair* $(L, r) = L^r$ *is a matrix with* $\sigma(L^r) = 0$, $\Omega(L^r) = \{r\}$, *and* $\delta(L^r) = 0$.

b) *If* $F_1, \ldots, F_n$, $n \geq 0$, *are matrices such that* $\Omega(F_i) \cap \Omega(F_j) = \emptyset$, *for* $i \neq j$ *and* $1 \leq i, j \leq n$, *then the set* $F = \{F_1, \ldots, F_n\}$ *is a matrix where:*

(i) $\sigma(\emptyset) = 0$, *for* $n = 0$, *and* $\sigma(F) = 1 + \sum_{i=1}^{n} \sigma(F_i)$, *for* $n > 0$;

(ii) $\Omega(F) = \Omega(F_1) \cup \ldots \cup \Omega(F_n)$;

(iii) $\delta(r) = m + 1$, *for any* $r \in \Omega(F_i)$, $1 \leq i \leq n$, *where* m *is the depth of* r *in* F_i.

According to this definition, the atomic parts of the matrices are ground literals, and in general, a matrix is a nested set of occurrences of literals.

Example 7.2.1 *Let us consider* $A = (a, b, c, d)$, *associated with the elements of the lattice FOUR (see figure 5.3), and let* $R = \{0, 1, 2, 3\}$ *be an alphabet of positions. Then* $L = (a, \perp, 0)$ *and* $\sim M = (c, f, 1)$ *are ground literals, while* $\{\{L^0\}, \{\sim M^1\}\}$ *and* $\{\sim M^0, \{\{L^1\}, \sim M^2\}\}$ *are matrices over* (A, R).

A matrix can also be viewed as a *tree*, where some leaves are associated with literals. Figure 2 presents the tree corresponding to the second matrix of the example above.

Definition 7.2.3 *Let* F *be a matrix and* $l, m \in \{0, 1\}$. *The set of* formulas $\tilde{F}$, *represented by* F *with respect to* (l, m), *is inductively defined as follows:*

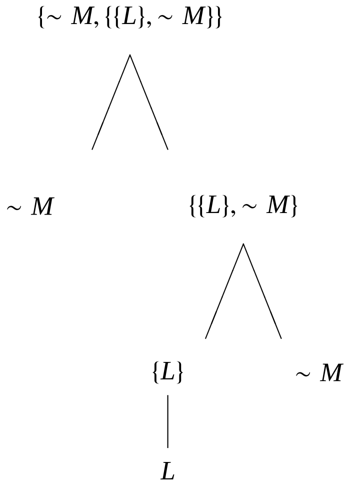

Figure 2: Representation of a matrix by a tree

(i) *If* F *is a literal* $F = L^r$ *and* $l = 0$, *then* $\tilde{F} = L$.

(ii) *If* F *is a literal* $F = L^r$ *and* $l = 1$, *then* $\tilde{F} =\sim L$.

(iii) *If* $F = \{F_1, \ldots, F_n\}$, $n \geq 0$, *and if* $m = 1$, *then* $\tilde{F} = \wedge(\tilde{F_1}, \ldots, \tilde{F_n})$, *where the* $\tilde{F_i}$ *are formulas represented by* F_i *with respect to* $(l, 0)$, $i = 1, \ldots, n$.

(iv) *If* $F = \{F_1, \ldots, F_n\}$, $n \geq 0$, *and if* $m = 0$, *then* $\tilde{F} = \vee(\tilde{F_1}, \ldots, \tilde{F_n})$, *where* $\tilde{F_i}$ *are formulas represented by* F_i *with respect to* $(l, 1)$, $i = 1, \ldots, n$.

Definition 7.2.4 *A formula* $\tilde{F}$ *is* positively represented *by a matrix* F *if it is represented by* F *with respect to* $l = m = 0$. $\tilde{F}$ *is* negatively represented *if it is represented by* F *with respect to* $l = m = 1$. *A* propositional formula *is any formula represented by some matrix. Formulas are also denoted by* D, E, F, *but this should not bring any confusion with the notation above.*

We use the following notation:

1. If $n = 0$, $\wedge(F_1, \ldots, F_n)$ is abbreviated by **T**, while $\vee(F_1, \ldots, F_n)$ is abbreviated by **F**.

2. If $n = 1$, $\wedge(F)$ and $\vee(F)$ are both abbreviated by F.

3. If $n \geq 2$, then $\wedge(F_1, \ldots, F_n)$ is a *conjunction*, while $\vee(F_1, \ldots, F_n)$ is a *disjunction*.

4. For any literal L, the formula $\neg^k L$ is called a *hyper-literal*. If L is a hyper-literal, then $\neg^k L = (a, \neg^k(\mu), p)$, where $\neg : \mathcal{T}_a \longrightarrow \mathcal{T}_a$ denotes some fixed function (that gives the meaning to the negation), and k is a multiplicity factor (a natural number).

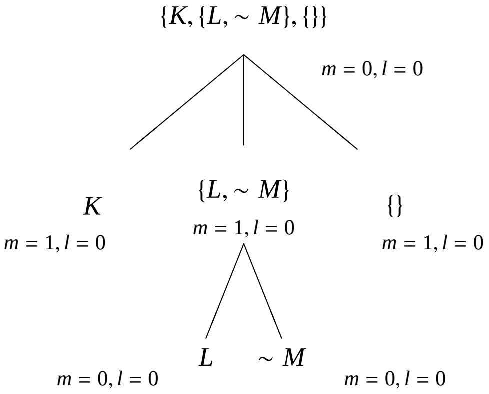

Figure 3: Positive representation of $F = \{K, \{L, \sim M\}, \{\}\}$

5. If $F = \wedge(F_1, \ldots, F_n)$ and $n \geq 0$, then $\sim F = \vee(\sim F_1, \ldots, \sim F_n)$.
6. If $F = \vee(F_1, \ldots, F_n)$ and $n \geq 0$, then $\sim F = \wedge(\sim F_1, \ldots, \sim F_n)$.
7. For any formula F, $\sim\sim F = F$.
8. Any formula $\sim F \vee G$ is abbreviated by $F \rightarrow G$.
9. Any formula $(F \rightarrow G) \wedge (G \rightarrow F)$ may be written as $F \leftrightarrow G$.
10. We also adopt the convention that the order of precedence decreases in the sequence $\sim, \wedge, \vee, \rightarrow, \leftrightarrow$. Parentheses, which are redundant given this convention, may be deleted.

Note that if F and G are formulas, then $\neg(F \rightarrow G)$ and $\neg \sim F$, for example, are not formulas.

According to the conventions above, every well-formed formula (defined in the standard way) determines a unique matrix. However, a matrix may represent more than one formula.

Example 7.2.2 *Let* $F = \{K, \{L, \sim M\}, \{\}\}$ *be a matrix. The tree in figure 3 is the positive representation of* F.

Example 7.2.3 *The formulas* $K \wedge L \rightarrow M$, $K \rightarrow \sim L \vee M$, $L \wedge K \rightarrow M$ *are all represented by* $\{\sim K^1, \sim L^2, M^3\}$.

The results presented in [44], chapter 2, remain applicable here, such as the following:

- If a formula $\tilde{F}$ is positively represented by a matrix F, then $\sim \tilde{F}$ is negatively represented by F.

- If two formulas $\tilde{F}_1$ and $\tilde{F}_2$ are positively represented by the same matrix F, then $\tilde{F}_1$ and $\tilde{F}_2$ are logically equivalent in the sense of annotated logics.

These results justify the use of matrices instead of formulas. The following example justifies the name *matrix*, that was first employed by Bibel (see [44]) and is also used here (see also [152]).

Example 7.2.4 *Let $\tilde{F}$ be the formula*

$$(K\wedge \sim L \rightarrow\sim N) \wedge M \wedge \neg L \rightarrow (\sim N\wedge \sim K)$$

where K, L, M and N are literals. If we put $\tilde{F}$ in the disjunctive normal form (as usual), we have:

$$(K\wedge \sim L \wedge N)\vee \sim M\vee \sim \neg L \vee (\sim N\wedge \sim K)$$

This formula can be represented by a bi-dimensional arrangement, where the literals, placed in a fixed column, are connected by '$\wedge$', while the columns are connected by '$\vee$', as follows:

$$F = \begin{bmatrix} K & & & \sim N \\ \sim L & \sim M & \sim \neg L & \\ N & & & \sim K \end{bmatrix}$$

With regard to semantics, let us first introduce the following definition:

Definition 7.2.5 *An* interpretation $\mathcal{M}$ *is a function that associates to each propositional symbol, an element of a chosen lattice. By denoting $\mathcal{M}(a) = \mu_a$, we can write $(a, b, c, \ldots) \mapsto (\mu_a, \mu_b, \mu_c \ldots)$.*

We can now define the 'truth value' $\mathcal{M}(F)$ of a matrix F as follows:

- If F is a literal $(a, \mu, 0)$, then $\mathcal{M}(F) = \mathbf{T} = \{\emptyset\}$ iff $\mathcal{M}(a) \geq \mu$; otherwise, $\mathcal{M}(F) = \mathbf{F} = \emptyset$.

- If F is a literal $(a, \mu, 1)$, then $\mathcal{M}(F) = \mathbf{F} = \emptyset$ iff $\mathcal{M}(a) \geq \mu$; otherwise, $\mathcal{M}(F) = \mathbf{T} = \{\emptyset\}$.

- If F is a matrix $F = \{F_1, \ldots, F_n\}$, $n \geq 0$, then $\mathcal{M}(F) = \bigcup_{k=1}^{n} \mathcal{M}(F_k)$ if $m = 0$, and $\mathcal{M}(F) = \bigcap_{k=1}^{n} \mathcal{M}(F_k)$ if $m = 1$.

We write $\mathcal{M}$ `sat` $\tilde{F}$ (and also $\mathcal{M}$ `sat` F) iff $\mathcal{M}(F) = \mathbf{T}$, for a matrix F that represents $\tilde{F}$.

Definition 7.2.6 *A matrix F is* valid *iff $\mathcal{M}(F) = \mathbf{T}$, for every interpretation. It is called* contradictory *iff $\mathcal{M}(F) = \mathbf{F}$, for every interpretation.*

Note that F is valid iff $\sim F$ is contradictory.

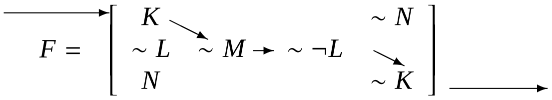

Figure 4: The path $\{K, \sim M, \sim \neg L, \sim K\}$ through the matrix F

Paths, Connections, and Validity

Definition 7.2.7 *A* path *through a matrix F is a set of occurrences of literals, defined as follows:*

1. *If $F = \emptyset$, then the only path through F is $\emptyset$.*
2. *If $F = L^r$, then the only path through F is the set $\{L^r\}$.*
3. *If $F = \{F_1, \ldots, F_m, F_{m+1}, \ldots, F_{m+n}\}$, $m, n \geq 0$, $m + n \geq 1$, for m literals $F_1, \ldots, F_m$ and for n matrices that are not literals $F_{m+1}, \ldots, F_{m+n}$, then for any matrix $E_i \in F_{m+i}$, and for any path p_i through E_i, $1 \leq i \leq n$, the set $\bigcup_{j=1}^{m} \{F_j\} \cup \bigcup_{i=1}^{n} p_i$ is a path through F.*

Example 7.2.5 *Let F be the matrix of the Example 7.2.4. A path through F is a path in the matrix from left to right, constrained to pass by the literals (that should be interpreted as 'gates'), as shown in Figure 4.*

Definition 7.2.8 *The two literals $L = (a, \mu, p)$ and $M = (a, \nu, q)$ are* complementary *iff:*

- *$p = 0$, $q = 1$ and $\nu \geq \mu$, or*
- *$p = 1$, $q = 0$ and $\mu \geq \nu$.*

Definition 7.2.9 Connections *are paths that have complementary literals as elements.*

Example 7.2.6 *Let $\mathcal{T}_a$ be the lattice FOUR, $\sim K = (a, t, 1)$, $\sim L = (a, f, 1)$, $M = (a, \top, 0)$, and let the matrix $F = \{\sim K, \sim L, M\}$. The (singleton) path through F has no complementary literals. In fact, neither $\sim K$ nor $\sim L$ are complementary to M. Alternatively, it suffices to see that $\top \nleq t$ and $\top \nleq f$, and hence, M cannot be connected by either $\sim K$ or $\sim L$.*

Definition 7.2.10 *We say that an ordered n-tuple $(\mu_1, \ldots, \mu_n)$, $n > 1$, of non-comparable elements of $\mathcal{T}_a$ is a* decomposition *of μ if $\mu = \sqcup\{\mu_1, \ldots, \mu_n\}$ and there are no non-comparable elements $\{\mu'_1, \ldots, \mu'_n\}$ of $\mathcal{T}_a$ such that $\mu'_i < \mu_i$ and $\mu = \sqcup\{\mu'_1, \ldots, \mu'_n\}$. We note that this definition is efficient for small discrete lattices.*

Example 7.2.7 *Consider the complete lattice FOUR. Then the only decomposition of the element $\top$ are the elements $\{t, f\}$, and hence, the matrix of the last example now is $F = \{\sim K, \sim L, \{M_1, M_2\}\}$, where $M_1 = (a, t, 0)$ and $M_2 = (a, f, 0)$. Note that K is complementary to M_1 and L is complementary to M_2. Alternatively, note that $t \leq t$ and $f \leq f$.*

Theorem 7.2.1 (Soundness and Completeness) *A matrix F is valid iff every path through F has a connection.*

Proof: Adapted from [44, pp. 30-31], by using induction on the size of σ in the matrix F. ▮

Checking the validity of a formula In our setting, the theorem proving technique consists in developing the following items:

a) First, we construct a matrix containing a goal and all the premises (or knowledge base).

b) We then check this matrix for validity.

Suppose that we have a set of formulas $\Gamma = \{F_1, \ldots, F_n\}$ and a query G. Then, to determine whether G is a semantic consequence of Γ, we should verify that the matrix provided by $(\bigwedge_{i=1}^{n} F_i) \to G$ is valid.

The paraconsistent component appears in this procedure given our definition of complementary literals. In this case, the existence of both a literal $L = (a, \mu, p)$ and its 'negation' $\neg L = (a, \neg(\mu), p)$ is not a sufficient condition to guarantee the existence of complementary literals in the path (see the definition 7.2.8). This illustrates the key ideas of the 'paraconsistent program' (see [74]). Note that to obtain a proof of the query, it is necessary that all paths of the matrix $\sim (\bigwedge_{i=1}^{n} F_i) \ \vee G$ have connections. This implies the existence of complementary literals in every path.

A case study Consider the construction of a simple medical system, aimed at diagnosing three diseases K, L and M. Let us suppose that there are two different symptoms, denoted by N and O. The intended usage of the system can be described as follows:

(i) The core part of the system is the knowledge provided by a doctor (DOC_1).

(ii) Before applying this knowledge to a specific patient, say Paul, the pathologists, X-ray technicians and other professionals who conduct medical tests on Paul add the results of these tests to the knowledge base.

(iii) To use the system, we submit a goal to the program similarly to what is done in PROLOG.

Usually, in the system, the main knowledge base described above is kept in one file, while each patient's records are maintained in a separate file (or possibly as a record in a given file). The main knowledge base and this file (or its record) are merged together to form an updated knowledge base. The latter is then used in the program that diagnoses the patient's disease.

We assume that the system is written as a finite set of annotated formulas over FOUR. Suppose also that DOC_1 provided the following five rules (formulas):

$$\begin{bmatrix} K_t & L_t & K_t & N_t & O_t & \sim N_t & K_t \\ \sim L_f & \sim K_f & \sim M_t & \sim K_t & \sim L_t & & L_f \end{bmatrix}$$

Figure 5: The matrix of the formula $F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge N_t \rightarrow (K_t \wedge L_f)$

(F_1) $K_t \rightarrow L_f$
(F_2) $L_t \rightarrow K_f$
(F_3) $K_t \rightarrow M_t$
(F_4) $N_t \rightarrow K_t$
(F_5) $O_t \rightarrow L_t$

Intuitively, the doctor is asserting that:

- An individual cannot have both diseases K and L. (F_1 and F_2)
- If an individual has the disease K, then he has the disease M. (F_3)
- If an individual has the symptom N, then he has the disease K. (F_4)
- If an individual has the symptom O, then he has the disease L. (F_5)

To exemplify the use and behavior of the program, we describe three situations. The first is similar to asking a query to a PROLOG program, while the other two explore the capacity of our method to handle inconsistencies:

Case 1: Suppose that the pathologist tell us that Paul was tested positively for the symptom N. And we want to know whether Paul has the disease K, but not L.

To answer this query, we must verify whether the matrix of the formula

$$F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge N_t \rightarrow (K_t \wedge L_f)$$

is valid.

First, we transform the formula into its disjunctive normal form, and then into its matrix form. The result of this process is shown in Figure 5. Now we must check whether all paths in this matrix are complementary. Note that the first path $\{K_t, L_t, K_t, N_t, O_t, \sim N_t, K_t\}$ has as complementary literals N_t and $\sim N_t$. And it is easy to see that all paths in this matrix have complementary literals. Since the matrix is valid, we conclude that Paul has the disease K, but not L.

Case 2: Suppose now that the pathologist determined that Paul was tested positively for symptoms N and O. And we want to know whether Paul has both diseases K and L.

To verify this query, we must check whether the matrix of the formula

$$F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge N_t \wedge O_t \rightarrow (K_t \wedge L_t)$$

is valid.

Following the same procedure used in the first case, it is easy to see that the matrix for this formula is valid. We conclude that Paul has both diseases K and L.

Note that this conclusion is in contradiction to what DOC_1 said: see the formulas F_1 and F_2 above. The reason for this conclusion is that the information given by the pathologist is in contradiction with that provided by the doctor. In a classical logic setting, this contradiction would make the knowledge base trivial (all formulas in the language could be derived from the knowledge base). In our case, the inconsistency is limited to the literals K and L; any atomic formula containing these literals can be proved.

Case 3: The previous example shows an inconsistent knowledge base, used to derive an inconsistent formula. The system, however, can handle inconsistencies in a non-trivial way. Let us exemplify this situation using the same set of symptoms as in Case 2 to determine whether Paul *does not have* the disease M. The latter condition can be written as M_f. So, the new situation is described by the following formula:

$$F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge N_t \wedge O_t \rightarrow M_f$$

Following the evaluation procedure once more, we verify that the matrix for this formula is *not valid*.

The last two queries show that our method can deal with inconsistencies in the knowledge base without every formula becoming derivable.

Implementation Issues In [166], an implementation of our method, using Standard ML Language [188], was described. But this process is not discussed here for reasons of space.

The ideas presented here can be used to model reasoning made by an intelligent agent that admits contradictory information. We believe that a system with this feature is more robust than the traditional ones. After all, the system keeps the inconsistency within a subset of its formulas, without affecting other parts of the knowledge base.

7.2.4 Inductive paraconsistent logic

Science does not use only deductions, of course. It is natural to ask then whether we can build 'non-classical inductive logics' on a pair with non-classical deductive logics, to develop systems especially in artificial antelligence (AI). John Pollock, who has been investigating several aspects of defeasible reasoning in AI, notes that:

> A common misconception about reasoning is that reasoning is deducing, and in good reasoning the conclusions follow logically from the premisses. It is now generally recognized both in philosophy and in AI that non-deductive reasoning is at least as common as deductive reasoning, and a reasonable epistemology must accommodate both. ([209])

The literature on non-deductive forms of reasoning presents various systems of computational tools devoted to non-monotonic and defeasible reasoning (see, e.g.,

[205] and [209], to mention just two works). A particularly difficult task for AI experts is to improve AI systems so that the latter can 'reason' and 'make inferences' from vague propositions, as we humans usually do. These propositions are such that we cannot, in general, assign with certainty one of the two truth-values, *true* or *false*. Frequently, our (human) reasoning is performed by attributing only a certain 'degree of confidence' as to whether the propositions in question are true or not. Suppose, for instance, that we are visiting a foreign country, and our informant gives us a vague information about the location of a certain building. (Perhaps our informant is also not sure about the right geography of the city.) We 'believe' in the information with a certain degree of confidence, and 'decide' the way to be taken.

We will outline now of a way of accommodating such 'degrees of confidence' in terms of paraconsistent logic. The result is a different process of dealing with vagueness, which, we believe, may be of interest to AI researchers. Our main motivation is, of course, to handle vague information mechanically. But, in this section, we will limit ourselves to the description of a vague inductive logic that, we hope, will be useful in the mechanical treatment of inductive information. Further developments should lead to the development of expert systems based on the proposal below. (The main ideas are from [98].)

First, we need to specify in what sense we are using the word 'inductive' in 'inductive reasoning'. We take as *inductive* any reasoning such that the truth of the premises does not entail necessarily the truth of the conclusion. Instead, the conclusion should be regarded as 'plausible' in light of the premises. There are ways of measuring this 'plausibility', but we need not discuss this issue here (for details, see [79]).

Our question then is: is there an 'inductive' paraconsistent logic? Positive answers were given, from different perspectives, in [174] and [98]. In the latter work (which will be our focus here), an 'inductive annotated system' was developed to deal with vagueness and degrees of confidence. The idea is that, when we face situations of vague information, such as the one about locating a building in a foreign city, we should opt for the more prudent alternative, according to a 'warning rule'. This rule tells us to be cautious in attributing degrees of confidence. In particular, we should not accept conclusions with degrees of confidence smaller than those of the premises. This is motivated by a rational stance, since a significant feature of rationality is the attempt to optimize our rational degrees of confidence in the propositions we are concerned with.

The proposal is to extend the common way of using arguments by accepting that some propositions are vague. For instance, 'Peter is smart' is a vague proposition. However, we still assign a degree of confidence in the truth of these propositions. In other words, we believe 'Peter is smart' with some degree of confidence. Peter's mother may have a great confidence in such a proposition, but his teacher may be not so confident. The degree of confidence can be interpreted as the amount of confidence someone has in the truth of a proposition.

To accommodate these two concepts, namely, vagueness and degree of confidence, we use the resources of annotated logics.

Let us call $\mathcal{I}_\tau$ a propositional logic whose language has the following categories of primitive symbols: (a) a countable set of propositional letters, which stand for propositions (we use $P, Q, \ldots$ as syntactic variables for propositions); (b) the elements

$\mu, \ldots, \mu_1, \ldots$ of a complete lattice τ ordered by $\leq$, which are called *the values of vagueness*; (c) the usual logical connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$), as well as (d) auxiliary symbols (parentheses).

The concept of *formula* of $\mathcal{I}_\tau$ is introduced in the following way:

(i) If P is a propositional letter and $\mu \in \tau$, then $P : \mu$ is a formula of $\mathcal{I}_\tau$ (atomic formula).

(ii) If α and β are formulas, then $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$ are formulas.[20]

(iii) Every formula is obtained only from the two clauses above.

Intuitively speaking, $P : \mu$ means that P is true with degree of vagueness μ. Note that we are assigning degrees of vagueness to atomic formulas only, and not to formulas in general. So, expressions such as:

$$((P : \mu_1) \vee (Q : \mu_2)) : \mu \tag{12}$$

are *not* well formed in our system.

Definition 7.2.11

(i) *If P is a propositional letter and $\mu \in \tau$, then:*

(a) $\neg^0 P : \mu$ *means* $P : \mu$

(b) $\neg^1 P : \mu$ *means* $\neg(P : \mu)$

(c) $\neg^k P : \mu$ *means* $\neg(\neg^{k-1}(P : \mu))$*, where k is a natural number, $k \neq 0$.*

(ii) *Let* $\sim: \tau \longrightarrow \tau$ *be a fixed mapping.*[21] *From now on, we will write* $\sim \mu$ *instead of* $\sim (\mu)$*. If $\mu \in \tau$, then:*

(a) $\sim^0 \mu$ *means* μ

(b) $\sim^1 \mu$ *means* $\sim \mu$

(c) $\sim^k \mu$ *means* $\sim (\sim^{k-1} \mu)$*, where $k \neq 0$ is a natural number.*

Expressions such as $P : \mu$ are called *annotated atoms*, while $\neg^k(\alpha : \mu)$ are *hyper-literals* of order k ($k \geq 0$). The other formulas are called *complex*.

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[20] $\alpha \leftrightarrow \beta$ is introduced in the standard way.

[21] The specific definition of this mapping depends on the particular application. For instance, if we take τ to be the unit interval $[0, 1] \subseteq \Re$ and $\sim (x) =_{\text{def}} 1 - x$, the introduction of 'fuzzy' ways of reasoning can be performed within the scope of annotated logics (see [116]).

Semantics Let τ be the complete lattice above with least element $\bot$ and greatest element $\top$. Let $h : \mathcal{P} \longrightarrow \tau$ be a mapping, called an *interpretation* of $\mathcal{I}_\tau$, where $\mathcal{P}$ is the collection of propositional letters of $\mathcal{I}_\tau$. The image of the proposition P by the mapping h will be denoted $P : \mu$, where $\mu \in \tau$. Informally speaking, as we noted, $P : \mu$ means that P is true with degree of vagueness μ. To each interpretation h, we associate a *valuation* $v_h : \mathcal{F} \longrightarrow \{0, 1\}$, where $\mathcal{F}$ is the collection of formulas of $\mathcal{I}_\tau$ defined above. Intuitively speaking, 1 and 0 stand for 'true' and 'false', respectively.

Particular applications may demand appropriate choices of the complete lattice. Here, to accommodate the case studies mentioned above, we will be concerned with a particular finite, linearly ordered set $\tau = \{\mu_1, \ldots, \mu_4\}$ (with $\mu_1 \leq \cdots \leq \mu_4$) to express the distinct degrees of vagueness of a proposition. But, clearly, the approach presented here is quite general.

Definition 7.2.12 *Let h and v_h be as above. Let P be a propositional letter, and let α and β denote formulas. Then:*

(i) $v_h(P : \mu) = 1$ *iff* $\mu \leq h(P)$.

(ii) $v_h(\neg^k(P : \mu)) = v_h(\neg^{k-1}(P :\sim \mu))$, *where* $k \neq 0$.

(iii) $v_h(\alpha \wedge \beta) = 1$ *iff* $v_h(\alpha) = v_h(\beta) = 1$.

(iv) $v_h(\alpha \vee \beta) = 1$ *iff* $v_h(\alpha) = 1$ *or* $v_h(\beta) = 1$.

(v) $v_h(\alpha \to \beta) = 1$ *iff either* $v_h(\alpha) = 0$ *or* $v_h(\beta) = 1$.

(vi) If α is a complex formula, then $v_h(\neg\alpha) = 1$ *iff* $v_h(\alpha) = 0$.

If $v_h(\alpha) = 1$, we say that v_h *satisfies* α, and that it does not satisfy α otherwise (that is, when $v_h(\alpha) = 0$). If Γ is a set of formulas, then we say that a formula α is a *semantic consequence* of (the formulas of) Γ, and write $\Gamma \models \alpha$, iff for every valuation v_h such that $v_h(\beta) = 1$ for each $\beta \in \Gamma$, $v_h(\alpha) = 1$. A formula α is *valid* iff $\emptyset \models \alpha$, and in this case we write $\models \alpha$.

As usual, we say that a valuation v_h is a *model* of a set of formulas Γ iff $v_h(\beta) = 1$, for every $\beta \in \Gamma$. In particular, v_h is a model of α iff $v_h(\alpha) = 1$. Other concepts, such as maximal non-trivial sets of formulas, are defined in the usual way.

The postulates of $\mathcal{I}_\tau$ If α, β, and γ are formulas and P is a propositional letter, then the postulates (axioms plus inference rules) of $\mathcal{I}_\tau$ are the following:[22]

(I1) All the postulates of classical positive logic.

(I2) If α and β are complex formulas, then the following is an axiom: $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$.

(I3) If α is complex, then $\alpha \vee \neg\alpha$ is an axiom.

(I4) If α is complex and β is an arbitrary formula, then $\alpha \to (\neg\alpha \to \beta)$ is an axiom.[23]

(I5) $P :\bot$ is an axiom.[24]

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[22] These postulates are adapted from [116] and [108].

[23] In other words, classical logic holds for complex formulas. The presence of inconsistencies will be allowed at the level of atomic formulas only (see [116]).

[24] The technical reason for using this axiom is that $v_h(P :\bot) = 1$ iff $h(\alpha) \geq 0$, which is always true.

(I6) If $\lambda \leq \mu$, then $P : \mu \rightarrow P : \lambda$

(I7) $\neg^k(P : \mu) \leftrightarrow \neg^{k-1}(P : \sim\mu)$, if $k \neq 0$.

(I8) Let α be an arbitrary formula. In this case, if $\alpha \rightarrow (P : \mu_i)$, $i \in I$, then $\alpha \rightarrow (P : \bigsqcup_{i \in I} \mu_i)$. If τ is a finite lattice, then this axiom may be replaced by the following one (see [116]):

$$P : \mu_1 \wedge \ldots \wedge P : \mu_n \rightarrow P : \bigsqcup_{i=1}^{n} \mu_i \tag{13}$$

The syntactic concepts of $\mathcal{I}_\tau$ are introduced in the standard way, as well as, in particular, the symbol of deduction $\vdash$ (see [116]).

We can prove the soundness and completeness of the logic $\mathcal{I}_\tau$ with respect to the semantics described in the previous section, and we will outline the result below. Let us first introduce a definition:

Definition 7.2.13 (Strong Negation)

$$\neg^* \alpha =_{\text{def}} \alpha \rightarrow ((\alpha \rightarrow \alpha) \wedge \neg(\alpha \rightarrow \alpha)) \tag{14}$$

It is easy to prove that $\neg^*$ has all the properties of classical negation. Hence, classical laws hold when $\neg^*$ is used instead of $\neg$ in the formulas of our system. For instance, *reductio ad absurdum*, $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg^*\beta) \rightarrow \neg^*\alpha)$, is a theorem of $\mathcal{I}_\tau$, and so is excluded middle, $\alpha \vee \neg^*\alpha$. Note that we can also show that if α is a complex formula, then $\neg\alpha \leftrightarrow \neg^*\alpha$ is valid (see the theorem below). However, this does not hold for formulas in general. For instance, if Q is a hyper-literal, then $\neg Q \leftrightarrow \neg^* Q$ is not valid in general (see [116]).

Other results are the following:

Theorem 7.2.2

(i) *If* $\Gamma, \alpha \vdash \beta$, *then* $\Gamma \vdash \alpha \rightarrow \beta$ *(deduction theorem).*

(ii) *If* $\Gamma \vdash \alpha$ *and* $\Gamma \vdash \alpha \rightarrow \beta$, *then* $\Gamma \vdash \beta$

(iii) $\alpha \wedge \beta \vdash \alpha$, $\alpha \wedge \beta \vdash \beta$, $\alpha, \beta \vdash \alpha \wedge \beta$

(iv) $\alpha \vdash \alpha \vee \beta, \beta \vdash \alpha \vee \beta$

(v) $\Gamma, \alpha \vdash \gamma$ *and* $\Gamma, \beta \vdash \gamma$, *then* $\Gamma, \alpha \vee \beta \vdash \gamma$ *(proof by cases)*

(vi) $\Gamma, \alpha \vdash \beta$ *and* $\Gamma, \alpha \vdash \neg^*\beta$, *then* $\Gamma \vdash \neg^*\alpha$ *(reductio ad absurdum)*

(vii) $\alpha, \neg^*\alpha \vdash \beta$, $\neg^*\neg^*\alpha \vdash \alpha$, $\alpha \vdash \neg^*\neg^*\alpha$

(viii) *If* α *is complex, then* $\neg^*\alpha \leftrightarrow \neg\alpha$.

(ix) $(\alpha : \mu_i)_{i \in I} \vdash \alpha : \bigsqcup_{i \in I} \mu_i$

(x) *If* $\Gamma \vdash \alpha$, *then* $\Gamma \models \alpha$ *(soundness theorem).*

To prove the completeness theorem, we need a few definitions and results.

Definition 7.2.14

(i) $\overline{\Gamma} =_{\mathsf{def}} \{\alpha : \Gamma \vdash \alpha\}$

(ii) Γ *is trivial iff* $\overline{\Gamma} = \mathcal{F}$, *where* $\mathcal{F}$ *is the set of formulas of* $\mathcal{I}_\tau$. *Otherwise,* Γ *is non-trivial*.

(iii) Γ *is inconsistent iff there exists* α *such that both* α *and* $\neg\alpha$ *belong to* $\overline{\Gamma}$. *Otherwise,* Γ *is consistent*.

(iv) Γ *is strongly inconsistent iff there exists* α *such that both* α *and* $\neg^*\alpha$ *belong to* $\overline{\Gamma}$. *Otherwise,* Γ *is strongly consistent*.

It is easy to see that Γ is strongly inconsistent iff it is trivial, and that Γ is strongly consistent iff it is non-trivial. Furthermore, for a suitable choice of τ, we can prove that there exist inconsistent but non-trivial sets of formulas, which are still not strongly inconsistent (see [104]). So, the logic $\mathcal{I}_\tau$ is paraconsistent. This means that there exist interpretations h and formulas α such that $v_h(\alpha) = v_h(\neg\alpha) = 1$. But we can also prove that for a certain τ, there are formulas α and interpretations h such that $v_h(\alpha) = v_h(\neg\alpha) = 0$. So, $\mathcal{I}_\tau$ is also a paracomplete logic. These results are treated in detail in several papers listed in the References (see, e.g., [116] and [104]).

Lemma 7.2.1 *Every non-trivial set of formulas is a subset of some maximal non-trivial set of formulas.*

Proof: See [116] and [104].▮

The completeness theorem results from the following lemma:

Lemma 7.2.2 *If* Γ *is a maximal non-trivial set of formulas, then its characteristic function* $\chi_\Gamma : \mathcal{F} \longrightarrow \{0, 1\}$ *is a model of* Γ*; that is, this mapping is a valuation such that* $\chi_\Gamma(\beta) = 1$, *for every* $\beta \in \Gamma$.

Proof: The idea is to define, for a given interpretation h, a valuation v_h, in such a way that the rules of $\mathcal{I}_\tau$ are 'preserved'. This means that, given Γ, we may define $h : \mathcal{P} \longrightarrow \tau$ such that for every proposition $P \in \mathcal{P}$,

$$h(P) =_{\mathsf{def}} \bigsqcup_i \{\mu_i : \mu_i \in \Gamma\} \tag{15}$$

It is not difficult to prove that the valuation generated by this interpretation coincides with the characteristic function χ_Γ.▮

As a consequence, we have the completeness theorem:

Theorem 7.2.3 *If* $\Gamma \models \alpha$, *then* $\Gamma \vdash \alpha$.

Proof: See [104] and [116].▮

Degrees of confidence We will sketch now a *theory of confidence*, which enables us to assign degrees of confidence to propositions, even to vague ones. Our degrees of confidence are in general only qualitative, characterized by the elements of an appropriate lattice with least and greatest elements. In abstract terms, degrees of confidence are elements of a lattice σ, and are assigned to the formulas of the language $\mathcal{I}_\tau$ when the propositional variables are interpreted as denoting vague statements (in the sense discussed above).

Let σ be a lattice with least and greatest elements, denoted respectively by $\bot'$ and $\top'$. The algebraic lattice operations are represented by $\sqcap$ and $\sqcup$, and the corresponding partial order by $\leq$. Let $\mathcal{F}$ be the set of formulas of the logic $\mathcal{I}_\tau$, and let α and β denote arbitrary annotated propositions. We then define the mapping $\mathsf{C} : \mathcal{F} \longrightarrow \sigma$, which satisfies the following postulates:

(C1) $\mathsf{C}(\alpha \wedge \neg^* \alpha) = \bot'$

(C2) $\mathsf{C}(\alpha \vee \neg^* \alpha) = \top'$

(C3) $\mathsf{C}(\bigvee_{i \in I} \alpha_i) \geq \bigsqcup_{i \in I} \mathsf{C}(\alpha_i)$, where I is finite.

(C4) $\mathsf{C}(\bigwedge_{i \in I} \alpha_i) \leq \sqcap_{i \in I} \mathsf{C}(\alpha_i)$, where I is finite.

(C5) If $\vdash \alpha \leftrightarrow \beta$, then $\mathsf{C}(\alpha) = \mathsf{C}(\beta)$.

Any mapping such as C is called a *confidence function*. Depending on the application under consideration, we can extend the postulates above. For instance, we can add the following ones: (i) if $\vdash \alpha \rightarrow \beta$, then $\mathsf{C}(\alpha) \leq \mathsf{C}(\beta)$, and (ii) $\mathsf{C}(\alpha) \sqcup \mathsf{C}(\neg^* \alpha) = \top'$.

Our proposal is to combine the concept of vagueness with that of confidence, that is, the logic $\mathcal{I}_\tau$ with the operator C. This is similar to the introduction of probability measures in a classical system of the probability calculus. Obviously, an 'algebra of confidence', as we suggest here, extends the classical case of subjective measure of probability, defined on Boolean algebras of propositions. Moreover, our procedure also encompasses Zadeh's theory of possibility (see [238] and [126]).

When the degrees of confidence reduce to strict degrees of belief (that is, to subjective probability), they can be handled in the standard way, following the rules of the Bayesian probability calculus. In the more general case, it is important to strengthen our system with additional rules. In particular, we add the following warning rule (note that, below, we write $P : \mu : \lambda$ for $\mathsf{C}(P : \mu) = \lambda$):[25]

[Warning rule]

$$\frac{P : \mu_i : \lambda_i\,,\ P : \mu_j : \lambda_j}{P : \mu_i \sqcap \mu_j : \lambda_i \sqcup \lambda_j} \quad \Gamma \tag{16}$$

Γ stands for the set of contextual conditions, which provide the basis for the application of the rule. These conditions are accepted as true.

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[25] Note also that, to state the rule, we used for simplicity $\sqcap$ and $\sqcup$ to denote the algebraic operations in both lattices τ and σ. This should not cause any confusion.

The warning rule has an intuitive appeal, and provides a tool to deal simultaneously with vagueness and degrees of confidence. Some inductive rules, such as those of induction by simple enumeration and analogy (see [81]), complemented by the approach to vagueness and the degrees confidence suggested here, are also useful to reinforce the logic we are trying to build. Similarly, the methods of the theory of possibility can be useful here.

We designate the resulting system of inductive logic by $\mathbf{I}_\tau$. Our main goal is to use an appropriate $\mathbf{I}_\tau$ to the mechanization of inductive inferences, say in robotics and in the theory of expert systems. Let us outline how this can be done by considering a simple example.

Suppose an insurance company needs to provide a way of classifying people into disjoint classes by age. For instance, the company needs to distinguish young from old people to differentiate among several prices for insurance premiums. The company will run a series of interviews to determine the age of the interviewees. Suppose also that the company decides to classify people in four different categories: (C_1) 'young' (less or equal to 25 years old); (C_2) 'not so young' (less or equal to 35 years old, and more than 25); (C_3) 'not so old' (between 35 and 45 years old), and (C_4) 'old' (more than 45 years old). Furthermore, suppose that the government decided to pay for insurance services also for people who don't have birth certificates, and in some cases, who don't know precisely their age. The hard and poor conditions of life these people face may confuse the interviewers if the latter try to attribute precise ages to the interviewees.[26]

As a result, the information provided by the interviewers may be considered, in some cases, as not completely precise. For instance, interviewer A may report to the company his belief that John seems to be 'not so old' (that is, it seems to the interviewer that John has an age between 35 and 45 years). Similarly, the interviewer may report that Paul looks 'not so young'. In other words, John and Paul were classified as satisfying, respectively, the predicates C_3 and C_2 above. But the interviewer is not completely sure about the correctness of the classification. If John and Paul have their birth certificates, in principle there would not be a problem as to how to deal with the information. After all, the interviewer's judgment about the age of each interviewee can be double checked.

Suppose, however, that neither John nor Paul have their birth certificates. How should we to deal with this situation? Since the attribution of age cannot be done with precision, the interviewer can adopt one of the following two alternatives: (1) to classify John as 'old' and Paul as 'not so young', or (2) to classify John as 'not so old' and Paul as 'young'. Several factors are involved in the choice between these two options. They include: the expertise of the interviewer (that is, her experience in the job); the interviewer's interest in the defense of a particular policy; or an 'external recommendation', for instance, the company's interest in classifying people in certain classes of ages so that the due premium prices are the highest possible (in this way, say, the company will receive more money from the government).[27]

Humans in general go around situations of this kind by accepting some (perhaps *ad hoc*) criterion. For example, they may decide what to do by adopting some rule

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[26] This situation is not uncommon in developing and poor countries.

[27] Of course, the situation could be precisely the opposite.

more or less arbitrarily, or by deciding on a case by case basis. But what happens if the situation should be handled by an expert system? In this case, is it possible to keep the system with the capacity of dealing with propositions to which different 'degrees of confidence' are attached to? Or is it possible to follow an insurance policy wherever it takes to express the 'confidence' in the information provided by an interviewer?

Let τ be a linearly ordered set $\tau = \{\mu_1, \ldots, \mu_4\}$ that is also a complete lattice (where $\mu_1(=\perp) \leq \ldots \leq \mu_4(= \top)$). Suppose that two interviewers[28] give as inputs $P : \mu_2$ and $P : \mu_3$, and suppose that P stands for 'John is old'. Then, according to the example we are discussing, $P : \mu_2$ means 'John is not so young', while $P : \mu_3$ says that 'John is not so old'. More precisely, the first interviewer has classified John as having an age between 25 and 35 years, while the second interviewer admitted that he is between 35 and 45 years old.

To attach a certain value to John's taxes, an expert system may follow a rule such as this: Let $C_1, \ldots, C_4$ be the classes of ages in the example above. *In case of doubt as to whether someone belongs to a class C_i or C_{i+1}, classify him/her as belonging to the class C_i.* So, in the case above, John would be considered as 'not so young', and then his due taxes would be supposed to be smaller than if he were classified as being 'not so old'.

The choice of $P : \mu_2$ can be interpreted as resulting from the application of the warning rule. After all, we assigned the greater degree of confidence to the vaguer proposition. In other words, recalling the convention made when we stated the warning rule, if we write $P : \mu_2 : \lambda_2$ for $\mathsf{C}(P : \mu_2) = \lambda_2$ and $P : \mu_3 : \lambda_3$ for $\mathsf{C}(P : \mu_3) = \lambda_3$, then we can say that from the 'premisses' $P : \mu_2$ and $P : \mu_3$, we arrived at the 'conclusion' $P : \mu_2$, which has greater degree of confidence. This can be expressed by saying that the 'conclusion' is $P : \mu_2 \sqcap \mu_3 : \lambda_2 \sqcup \lambda_3$. (Note that τ is a linear lattice in which $\mu_2 \sqcap \mu_3 = \mu_2$.)

Roughly speaking, the warning rule insists that an expert system, elaborated to accommodate vagueness and equipped with degrees of confidence, when faced with situations such as the one considered, should opt for the more prudent option. This is a perfectly rational move. After all, as noted above, rationality involves the attempt to optimize our rational degrees of confidence in the propositions under study, but with the caution of not taking conclusions with degrees of confidence smaller than those attributed to the premises.

7.3 Foundations of physics

7.3.1 Logic and physics

During the International Congress of Mathematicians, held in Paris in 1900, David Hilbert presented a list of 23 Problems of Mathematics that, in his view, should occupy the efforts of mathematicians in the following century. To solve one of these problems was to achieve something extraordinary in mathematics, and several Fields medals were awarded for this kind of effort. The sixth problem of Hilbert's celebrated list dealt with the axiomatization of physical theories. Hilbert proposed 'to treat in the

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[28] This number can be generalized, of course.

same manner [as Hilbert himself had done with geometry], by means of axioms, those physical sciences in which mathematics plays an important part' ([147]).

In the 20th century, much was done in this direction, continuing the efforts already developed in the previous century. There have also been a parallel development of logic in the 20th century, and the development of non-classical systems, linked to some philosophical views about science and the presentation of scientific theories, such as that one of the logical empiricists, with emphasis in logic and language. These developments forced philosophers to acknowledge that underlying the axiomatic version of a scientific theory, there are also logical postulates (often of a higher-order logic kind), which provide the grounds for the theory's deductive and mathematical counterparts. Roughly speaking, we can think as 'logical' those postulates of first-order logic (with identity), while the 'mathematical' postulates can be those of a set theory, such as Zermelo-Fraenkel. (Of course, alternative approaches can also be developed, but we will not discuss this point here.) In this framework, logic plays an important role, and if there are reasons to suspect that some logical system, other than classical logic, should be used to axiomatize a certain domain, its details must be made explicit.

For example, discussions about the relationship between logic and quantum physics tend to relate the subjects with so-called 'quantum logics'. This is a field that has its 'official' birth in Birkhoff and von Neumann's well-known paper from 1936. This is completely justified, for this fundamental work led to the development of a new field of research in logic. Today, there are various 'quantum logical systems', including some paraconsistent quantum logics (see [118], [119], and [120]). However, these systems have been studied especially as pure mathematical systems, far from applications to the axiomatization of the microphysical world and from the insights of the forerunners of quantum mechanics.

Of course, the axiomatization of a given empirical theory is not always completely determinate, and the need for a logic distinct from the classical as the underlying logic of quantum theory is still an open question. The axiomatic basis of a scientific theory depends on several aspects of the theory, some more explicit than others, that are needed to take into account the theory's structure. For instance, Ludwig studies an axiomatization of quantum mechanics based on classical logic (see [176]). All such approaches, from those that use a paraconsistent logic, or some other kind of logic, to those of Ludwig, are in principle acceptable. After all, they provide different perspectives to the same domain of discourse, and such different 'perspectives' of a domain of science may demand distinct logical apparatuses. This is a philosophical stance radically different from the classical, and it is against the idea that there is only one logic – classical logic.

As we noted in the Introduction, the possibility of using non-standard logical systems in the foundations of physics – and, more generally, in the foundations of science – does not entail that classical logic is wrong, or that quantum theory, in particular, *needs* another logic. Physicists are likely to continue to use classical, informal logic in the near future. But we should realize that other forms of logic may help us to understand better certain aspects of the quantum world that may not be so easily treated by classical means. (An interesting example is the concept of complementarity, which can be studied very naturally in a non-classical context.) Of course, only in the future it will be decided (if at all) whether the use of non-classical systems in science should (or should not) be preferred to the development of theories in a ‘classical’ framework. Clearly, any such decision will involve significant pragmatic factors.

To summarize our proposal, we do not believe that there is only one ‘true logic’, and we think that distinct logical systems systems, such as those provided by paraconsistent logic, are useful to approach different aspects of such wide domains of knowledge, such as quantum theory. The important point is scientists should be open to the justifiable revision of concepts (including logical ones). This point was very lucidly emphasized by Niels Bohr, who wrote:

> For describing our mental activity, we require, on the one hand, an objectively given content to be placed in opposition to a perceiving subject, while, on the other hand, as is already implied by this assertion, no sharp separation between object and subject can be maintained, since the perceiving subject also belongs to our mental content. From these considerations, it follows not only the relative meaning of every concept, or rather of every word (the meaning depending upon our arbitrary choice of viewpoint), but also that we must, in general, be prepared to accept the fact that a complete elucidation of one and the same object may require diverse points of view that defy a unique description. Indeed, strictly speaking, the conscious analysis of any concept stands in a relation of exclusion with its immediate application. The necessity of invoking a complementarity, or reciprocal, mode of description is perhaps most familiar to us from psychological problems. In opposition to this, the feature that characterizes the so-called exact sciences is, in general, the attempt to obtain uniqueness by avoiding all reference to the perceiving subject. This effort is found most consciously, perhaps, in the mathematical symbolism that presents for our contemplation an ideal of objectivity for whose achievement scarcely any limits are set, as long as we remain within a self-contained field of applied logic. In natural sciences, however, there can be no question of a strictly self-contained field of application of the logical principles, since we must continually expect the appearance of new facts, the inclusion of which within the boundaries of our earlier experience may require a revision of our fundamental concepts. (*ibid.*)

7.3.2 A case involving paraconsistency

In this section, we will discuss, as an illustration, how paraconsistent logic can be used to approach what we call ‘complementarity theories’. These theories are based on Bohr’s view on complementarity, a concept he introduced in his famous ‘Como Lecture’, in 1927, although the basic ideas go back to 1925. The consequences of his views were fundamental, particularly to the development of the Copenhagen interpretation of quantum mechanics, and they constitute, as is largely recognized in the literature, as one of the most significant contributions to the development of quantum theory. However, we are not making an exegesis of Bohr’s ideas, for, as is well known, they are quite controversial.[29] Rather, we will use a particular understanding of the notion of comple-

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[29] For references and further discussion of this particular point, see [43] and [99].

mentarity to provide the basis for defining a general class of theories (C-theories). As will become clear, the logic of such theories is a particular paraconsistent logic, called *paraclassic*.

Roughly speaking, we say that a theory T admits a complementarity interpretation, or that T is a C-theory, if T encompasses 'true' formulas α and β that are 'mutually exclusive', in the sense that their conjunction yields a contradiction if classical logic is applied. More precisely, let $\vdash$ be the symbol of deduction of classical logic. We say that T is a C-theory if there exists γ such that if α and β are complementary, we have (in T) $\alpha, \beta \vdash \gamma \wedge \neg\gamma$. Of course, if the underlying logic of T is classical logic, since T involves complementary propositions, T is contradictory, or inconsistent.

Let us call C an axiomatized system for the classical propositional calculus (the developments presented here can be extended to quantification). The concept of deduction in C is the standard one. We use the symbol $\vdash$ to represent deductions in C. Furthermore, the formulas of C are denoted by Greek lowercase letters, while Greek uppercase letters stand for sets of formulas. The symbols $\neg$, $\rightarrow$, $\wedge$, $\vee$ and $\leftrightarrow$ have their usual meanings, and standard conventions in writing formulas will be also assumed without further comments.

Definition 7.3.1 *Let Γ be a set of formulas of C, and let α be a formula (of the language of C). Then we say that α is a (syntactic) P-consequence of Γ, and write $\Gamma \vdash_{\mathsf{P}} \alpha$, if and only if*

(P1) *$\alpha \in \Gamma$, or*

(P2) *α is a classical tautology, or*

(P3) *there exists a consistent (according to classical logic) subset $\Delta \subseteq \Gamma$ such that $\Delta \vdash \alpha$ (in classical logic).*

We call $\vdash_{\mathsf{P}}$ the relation of P-*consequence*.

Definition 7.3.2 P *is the logic whose language is that of* C *and whose relation of consequence is that of* P*-consequence. This logic will be called paraclassic.*

It is immediate that, among others, the following results can be proved:

Theorem 7.3.1

1. *If α is a theorem of the classical propositional calculus* C *and if Γ is a set of formulas, then $\Gamma \vdash_{\mathsf{P}} \alpha$; in particular, $\vdash_{\mathsf{P}} \alpha$.*

2. *If Γ is consistent (according to* C*), then $\Gamma \vdash \alpha$ (in* C*) iff $\Gamma \vdash_{\mathsf{P}} \alpha$ (in* P*).*

3. *If $\Gamma \vdash_{\mathsf{P}} \alpha$ and if $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathsf{P}} \alpha$. (The notion of* P*-consequence is monotonic.)*

4. *The notion of* P*-consequence is recursive.*

5. *Since the theses of* P *(that is, valid formulas of* P*) are those of* C*,* P *is decidable.*

Definition 7.3.3 *A set of formulas Γ is* P-*trivial iff* $\Gamma \vdash_{\mathsf{P}} \alpha$*, for every formula* α*. Otherwise,* Γ *is* P-*non-trivial. (Similarly, we define the concept of a set of formulas being trivial in* C*.)*

Definition 7.3.4 *A set of formulas* Γ *is* P-*inconsistent if there exists a formula* α *such that* $\Gamma \vdash_{\mathsf{P}} \alpha$ *and* $\Gamma \vdash_{\mathsf{P}} \neg\alpha$*. Otherwise,* Γ *is* P-*consistent.*

Theorem 7.3.2

1. *If* α *is an atomic formula, then* $\Gamma = \{\alpha, \neg\alpha\}$ *is* P-*inconsistent, but* P-*non-trivial.*
2. *If the set of formulas* Γ *is* P-*trivial, then it is trivial (according to classical logic). If* Γ *is non-trivial, then it is* P-*nontrivial.*
3. *If* Γ *is* P-*inconsistent, then it is inconsistent according to classical logic. If* Γ *is consistent according to classical logic, then* Γ *is* P-*consistent.*

A semantic analysis of P, for instance a completeness theorem, can be obtained without difficulty, as indicated in [105]. Note that the set $\{\alpha \wedge \neg\alpha\}$, where α is a propositional variable, is trivial according to classical logic, but it is not P-trivial. However, we are not suggesting that complementary propositions should be understood necessarily as pairs of contradictory sentences. This is made clear by the following definition:

Definition 7.3.5 (Complementarity Theories or C_{mp}-theories) *A* C*-theory is a set of formulas* T *of the language of* C *(the classical propositional calculus) closed by the relation of* P-*consequence, that is,* $\alpha \in T$*, for any* α *such that* $T \vdash_{\mathsf{P}} \alpha$*. In other words,* T *is a theory whose underlying logic is* P*. A* C_{mp}*-theory is a* C*-theory that satisfies additional meaning postulates (that are introduced in the particular context in which the theory is used).*

Of course, the definition of a C_{mp}-theory is vague. But, in each context of use, what a C_{mp}-theory amounts to is clear. Moreover, note also that if a meaning postulate introduces restrictions in the acceptable statements of the theory, the hypotheses and axioms used in deductions must satisfy such restrictions as well. For instance, if a meaning postulate of a theory T is formulated as Heisenberg's uncertainty principle, this will impose obvious restrictions to certain statements of T.

Theorem 7.3.3 *There exist* C*-theories and* C_{mp}*-theories that are inconsistent, although they are* P-*non-trivial.*

Proof: Immediate consequence of Theorem 5.1.2. ▮

Finally, we state a result (Theorem 7.3.4), whose proof is an immediate consequence of the definition of P-consequence. However, before stating the theorem, let us introduce a definition:

Definition 7.3.6 (Complementary Propositions) *Let* T *be a* C_{mp}*-theory (in particular, a* C*-theory), and let* α *and* β *be formulas of the language of* T*. We say that* α *and* β *are* T*-complementary (or simply complementary) if there exists a formula* γ *of the language of* T *such that:*

1. $T \vdash_{\mathsf{P}} \alpha$ and $T \vdash_{\mathsf{P}} \beta$
2. $T, \alpha \vdash_{\mathsf{P}} \gamma$ and $T, \beta \vdash_{\mathsf{P}} \neg\gamma$ (in particular, $\alpha \vdash_{\mathsf{P}} \gamma$ and $\beta \vdash_{\mathsf{P}} \neg\gamma$).

Theorem 7.3.4 *If α and β are complementary theorems of a C_{mp}-theory T, and $\alpha \vdash_{\mathsf{P}} \gamma$ and $\beta \vdash_{\mathsf{P}} \neg\gamma$, then in general $\gamma \wedge \neg\gamma$ is not a theorem of T.*

Proof: Immediate, as a consequence of Theorem 5.1.2. ▮

In other words, T is inconsistent from the point of view of classical logic, but it is P-non-trivial.

It should be emphasized, once again, that our way of characterizing complementarity does not mean that complementary propositions are always contradictory, for α and β above are not necessarily the negation of each other. However, since they are complementary propositions, we can derive from them (in classical logic) a contradiction. For example, 'x is a particle' is not the direct negation of 'x is a wave', but 'x is a particle' *entails* that 'x is not a wave'. This reading of complementarity as not indicating strict contradiction is in accordance with Bohr's view. As he notes (see [99]):

> When we consider the well-known paradoxes that are encountered in the application of quantum theory to atomic structure, it is essential to remember, in this connection, that the properties of atoms are always obtained by observing their reactions under collisions or under the influence of radiation, and that the (...) limitation on the possibilities of measurement is directly related to the apparent contradictions that have been revealed in the discussion of the nature of light and of material particles. To emphasize that we *are not concerned here with real contradictions*, the author [Bohr himself] introduced in an earlier article the term 'complementarity'. [The italics is ours.]

Let us give a simple example of a situation involving a C_{mp}-theory. Suppose that our theory T is a fragment of quantum mechanics admitting Heisenberg's relations as a meaning postulate, and having as its underlying logic paraclassic logic. If α and β are two incompatible propositions according to Heisenberg's principle, we can interpret this principle as implying that α entails $\neg\beta$ (or that β entails $\neg\alpha$). So, even if we add α and β to T, we will be unable to derive, in T, $\alpha \wedge \beta$. Analogously, Pauli's exclusion principle has a similar interpretation to Heisenberg's.

As we have already noted, the basic feature of C_{mp}-theories is that, to make P-inferences, we suppose that some sets of statements we handle are consistent. In other words, C_{mp}-theories are closer to those theories scientists *actually* use in their everyday activity than those theories with the classical concept of deduction. That is, paraclassic logic – and paraconsistent logics in general – seem to fit more accurately the way scientists reason when they state and articulate their theories.

7.3.3 Generalization: the paralogic associated with a given logic

The technique used above to define the paraclassic logic associated with classical logic can be generalized to other logics $\mathcal{L}$. (This includes logics that have no negation symbol; but we will not discuss this case here.) The same point also applies to the concept of a C_{mp}-theory. More precisely, let $\mathcal{L}$ be a logic. It can be seen as a pair $\mathcal{L} = \langle \mathcal{F}, \vdash \rangle$, where $\mathcal{F}$ is a set of formulas of $\mathcal{L}$, and $\vdash \subseteq \mathcal{P}(\mathcal{F}) \times \mathcal{F}$ is the deduction relation of $\mathcal{L}$ (satisfying certain postulates depending on the particular logic $\mathcal{L}$). We define the $\mathsf{P}_{\mathcal{L}}$-logic associated with $\mathcal{L}$ (the 'paralogic' associated with $\mathcal{L}$) as follows:

Let $\mathcal{L}$ be a logic, which may be classical logic, intuitionistic logic, some paraconsistent logic or, in principle, any other logic. For simplicity, we suppose that the language of $\mathcal{L}$ has a symbol for negation, $\neg$. Then:

Definition 7.3.7 *A theory based on $\mathcal{L}$ (an $\mathcal{L}$-theory) is a set of formulas Γ of the language of $\mathcal{L}$ closed under $\vdash_{\mathcal{L}}$ (the symbol of deduction in $\mathcal{L}$). In other words, $\alpha \in \Gamma$, for every formula α such that $\Gamma \vdash_{\mathcal{L}} \alpha$.*

Definition 7.3.8 *An $\mathcal{L}$-theory Γ is $\mathcal{L}$-inconsistent if there exists a formula α of the language of $\mathcal{L}$ such that $\Gamma \vdash_{\mathcal{L}} \alpha$ and $\Gamma \vdash_{\mathcal{L}} \neg\alpha$, where $\neg\alpha$ is the negation of α. Otherwise, Γ is $\mathcal{L}$-consistent.*

Definition 7.3.9 *An $\mathcal{L}$-theory Γ is $\mathcal{L}$-trivial if $\Gamma \vdash_{\mathcal{L}} \alpha$, for every formula α of the language of $\mathcal{L}$. Otherwise, Γ is $\mathcal{L}$-non-trivial.*

We can now define the $\mathsf{P}_{\mathcal{L}}$-logic associated with $\mathcal{L}$, whose language and syntactic concepts are those of $\mathcal{L}$. The only exception is the concept of deduction, which is introduced as follows: We say that α is a $\mathsf{P}_{\mathcal{L}}$*-syntactic consequence* of a set of formulas Γ, and write $\Gamma \vdash_{\mathsf{P}_{\mathcal{L}}} \alpha$, if and only if:

(1) $\alpha \in \Gamma$, or

(2) α is a provable formula of $\mathcal{L}$ (that is, $\vdash_{\mathcal{L}} \alpha$), or

(3) there exists $\Delta \subseteq \Gamma$ such that Δ is $\mathcal{L}$-non-trivial, and $\Delta \vdash_{\mathcal{L}} \alpha$.

As an example, we can consider the paraconsistent calculus C_1 as our logic $\mathcal{L}$. Then the paralogic associated with C_1 is a kind of 'para-paraconsistent' logic.

As a final remark, note that, sometimes, given a paraclassic theory T such that $T \vdash_{\mathsf{P}} \alpha$ and $T \vdash_{\mathsf{P}} \neg\alpha$, there exist *appropriate* propositions β and γ such that T can be replaced by a classical consistent theory T' in which $\beta \to \alpha$ and $\gamma \to \neg\alpha$ are theorems. If this happens, the logical difficulty can be, in principle, solved, and classical logic can be maintained.

7.4 Morality and Law

In its standard form, deontic logic can be taken as a kind of classical modal logic with the introduction of deontic operators: O for obligatory, P for permitted, V for prohibited, and F for indifferent. One of the interesting problems about deontic logic is that standard systems do not enable the existence of 'true' moral dilemmas. The latter are expressed by formulas of the form $O\alpha \wedge O\neg\alpha$, stating that both α and its negation are obligatory. This is due to the fact that, in standard deontic systems, $O\alpha \wedge O\neg\alpha \to O\beta$ is a theorem. Hence, a moral dilemma trivializes the whole system, in the sense that all formulas become obligatory. Other problems about standard deontic logics are created by Ross' paradox and the Good Samaritan paradox (see [218]).

A standard solution to the problem about the non-existence of moral dilemmas is simply to maintain that such dilemmas really cannot exist. However, for some philosophers, they are genuine possibilities (see [87] for references). To accommodate this view, it is crucial to make room for moral dilemmas, but obviously without deontic triviality. Paraconsistent logics are excellent candidates for doing this.

The obligatory operator – as well as the other mentioned above – have various interpretations, the most common are in terms of *moral* obligations (and moral prohibitions etc.) and *legal* obligations (legal prohibitions etc.). If we write O_l and O_m for the legal and the moral obligations, respectively (the same notation applies to the other operators), it is important to study the principles $O_l\alpha \rightarrow P_m\alpha$ and $O_m\alpha \rightarrow P_l\alpha$, which seem to be quite 'natural'. To study these (and related) principles, some deontic paraconsistent logics have been introduced (see [216]).

7.5 Philosophical significance of paraconsistent logic

After discussing the formal details of several paraconsistent systems, we can now examine some philosophical issues raised by the various constructions above as well as by paraconsistency more generally. Several issues need to be addressed. In particular, the issues of whether there is a true paraconsistent logic, and how to choose between such logics should be examined. In our view, the choice between such logics is ultimately a pragmatic and context-dependent matter, largely dependent on the details of the applications at hand. We will develop these points below.

The constructions we have just presented make clear that, at least at the level of mathematics, there are inconsistent but non-trivial theories. In other words, there are theories T in which both a formula α and its negation $\neg\alpha$ are theorems of T, and some formula β of T is *not* a theorem. In our view, this fact provides an important support for paraconsistency, since it shows that the attempt at accommodating inconsistencies by devising appropriate inconsistent but non-trivial theories is by no means empty or unrealizable. On the contrary, it provides a distinctive perspective on the issues under consideration. Instead of retaining classical logic, and avoiding the inconsistency by rejecting one or another of the premises which generate it – making more or less *ad hoc* moves – we retain the inconsistency, change the underlying logic to a paraconsistent one, and study the properties of the 'inconsistent object' so 'generated'. The important feature, as the paraconsistent set theory discussed above shows, is that these 'inconsistent objects' have certain determined properties and lack others: it is simply *not* the case that everything goes with regard to them. So, as opposed to what happens in the case of classical logic, there is a whole new domain of investigation determined by the formulation of paraconsistent logic: the domain of the inconsistent.

Now, the issue arises as to the status of the resulting inconsistent set theory: is it *true*? Well, can we say that there are true contradictions? The answer depends on several considerations. (1) What is the notion of truth used in this context? (2) What kinds of objects are we considering (only mathematical objects or do we include physical objects as well)? (3) What notion of existence is assumed? And how are ontological commitments to be spelled out? Of course, the examination of these issues involves particular philosophical moves, and we cannot do more here than consider them in fairly general terms. But we hope to say enough to make clear the approach we favor.

According to some authors, the answer to the question *Are there true contradictions?* is affirmative (see [212]). The examples given by Priest are the logical and semantic paradoxes, statements about moving objects (objects subject to change), and certain views in the foundations of mathematics. In order to articulate an 'ontic' or 'realist' view about true contradictions, Priest advocates (i) a strong notion of truth – truth *simpliciter* understood in the correspondence sense – (ii) a classical view about existence (as the range of bound variables), and (iii) an extended claim as to the domain of his theory – which incorporates both mathematical and physical objects. (Of course, in order to avoid triviality, Priest adopts a paraconsistent logic: see his system LP discussed in [212].) So, Priest's approach countenances classical views about truth and existence, and applies them to a wide-ranging domain. In our view, Priest's commitment to several doctrines is by no means fortuitous: in order to be adequately accommodated, inconsistencies require a whole package of logical and philosophical doctrines – indeed, a whole research program is involved. Of course, there are stronger and weaker programs; some are closer to classical proposals, some farther away.

It seems to us that, in retaining classical notions of truth and existence, Priest's proposal ends up committed to strong metaphysical views. Given the use of truth in the correspondence sense, and the claim that our assertions about the world (be it the 'empirical' or the 'mathematical' world) are to be true, Priest's view is *ipso facto* committed to all objects that are posited by these claims. In particular, his proposal seems to be committed to 'inconsistent objects' in the physical world: the objects to which our inconsistent but true physical theories refer. But how can their existence be established?

The argument to this effect assumes, of course, the classical account of ontological commitment: we are ontologically committed to whatever our bound variables range over. And in the case of inconsistent theories, this criterion leads us to postulate objects that both have and lack a given property (for instance, the liar sentence is both true and false, Russell set is both a member of itself and it isn't etc.). And the same goes for theories about the physical world.

In our view, this argument is not so conclusive. First, this criterion of ontological commitment is not independent of particular philosophical assumptions. It comes as part of a philosophical program – indeed, Quine's view – and it has built into it, as it were, a given logic: *classical* first-order logic. It goes without saying that, as such, it is at odds with Priest's own dialetheic approach, in which a paraconsistent logic is advocated. Moreover, Quine's criterion is *not* independent of logic: if we change the underlying logic of a given theory, we may change the entities we are quantifying over as well. This can be seen in several ways. For example, consider that we move to second-order logic. We are then allowed to quantify over properties and relations. As the result of second-order logic's strong expressive power, several mathematical theories can be better formulated in this setting (in particular, as is well known, arithmetic and analysis are categorical in a second-order context). Because of this, several nominalist proposals (such as those developed in [129] and [145]) have adopted second-order logic as part of their nominalization strategies of science and mathematics. The idea is that, by increasing the strength of the logic, we can decrease our ontological commitments. Secondly, using paraconsistent logic, we are allowed to quantify over certain constructions (such as Russell set) that are impossible in classical logic, given the latter's identification of inconsistency and triviality.

Our point here is that Quine's slogan – to be is to be the value of a variable – can only have any force once a particular logic is admitted. Quine knows that, of course. The problem is that his view assumes a logic (classical fist-order logic) that is not the most adequate to deal with inconsistencies in a heuristically fruitful way.

We suggest a different way of addressing the inconsistency issue. We may well explore the rich representational devices allowed by the use of paraconsistency in inconsistent domains, but withholding any claim to the effect that there are 'inconsistent objects' in reality. Whether the world is indeed 'inconsistent' – assuming there is a sensible formulation of this claim – is something we would rather be *agnostic* about. Just as empiricists (such as van Fraassen [236]) are agnostic about the existence of unobservable entities in science, we are agnostic about the existence of true contradictions in nature. And one of the reasons in support of this claim is an *underdetermination* argument. Given the hierarchy of paraconsistent C-logics discussed above, there are always infinitely many paraconsistent logics that can be used to accommodate a given 'phenomenon' – whether it is an 'inconsistent' reasoning or an 'inconsistent' theory (see [113]). Which of these paraconsistent logics reflects *the* logic of the world? There is no argument based on purely observational terms that could settle the issue. We can select, of course, one of these logics on *pragmatic* grounds, but these grounds are certainly not enough to establish a substantive claim about the world. For instance, if one of these logics makes the modeling of the inconsistency in question easier, why should this be taken as a reason for this logic to be *true*? Simplicity may well be a sensible criterion to adopt on pragmatic grounds, but the claim that a logic selected on this basis is (likely to be) true, is to confuse pragmatic and epistemic considerations. Why should the world conform to our cognitive limitations? Of course, it might well do. But to establish this claim demands an argument that goes beyond what we observe: it requires a metaphysical commitment to the simplicity of reality. And, to a certain extent, this is as 'strong' as the claim that there are true contradictions. After all, both make substantial assertions about the world that transcend empirical observation. Both are metaphysical claims.

In our view, an alternative program of interpretation of inconsistencies can be devised in which no commitment to this kind of metaphysics is required. The idea is first to avoid the claim that inconsistent theories are *true*; they are *partially true* (or *quasi-true*) at best. As discussed above, the notion of partial truth receives a formal treatment. But for our needs, it suffices to recall that a sentence α is partially true if it models adequately only part of a given domain D, leaving open the 'complete' description of the latter. (Of course, in a precise sense, α is consistent with a true description of D.) In claiming that with regard to inconsistent theories all we need is to determine their partial truth, we are in position to provide a 'formal underpinning' to our agnosticism with regard to true contradictions. We are suggesting a change in the notion of truth to the weaker notion of partial truth, withholding then the commitment to 'inconsistent objects'. After all, several ($\mathfrak{A}$-normal) structures describe the domain under consideration, and inconsistent objects are not countenanced in all of them. [30] Moreover, we also suggest a revision of Quine's slogan about ontological commitment, making explicit its dependence on the underlying logic. In this way, it becomes clear that this slogan is *not* the only criterion to adjudicate between alternative logics.

However, if we end up not being committed to an ontology of actual inconsistent physical objects, are we committed to inconsistent *mathematical* entities? This depends, of course, on how we interpret the relevant mathematical theory. Does it provide a *true* description of the inconsistent mathematical 'world'? Again, if all we claim is to have described this 'world' in at best partially true terms, we will not be committed to those entities. There is, of course, a whole story to be told here, but for the present purposes, it suffices to note that it is possible to provide an entirely syntactic formulation of paraconsistent set theory, in which various inconsistent theories can be embedded, such that the only commitment is to a countable language. Thus, in a certain sense, we do not require a special commitment to mathematical objects either.

So, the 'package' we suggest to accommodate inconsistencies is characterized by (1) the claim that inconsistent theories are *partially true* at best; (2) an agnosticism with regard to the existence of true contradictions and a nominalism about inconsistent mathematical entities; and (3) a re-evaluation of Quine's view about ontological commitment, emphasizing its dependence on the underlying logic.

In our view, the striking feature of this 'package' is its logical pluralism, on the one hand, and the fact that it allows us to make sense of the several uses and applications of paraconsistent logic with no commitment to actual 'inconsistent objects'. The logical pluralism derives from claim (3) above. Depending on the domain we are considering, different kinds of logic may be appropriate. For instance, if we want to model constructive features in mathematical reasoning, an intuitionistic logic is the best alternative; if we are concerned with inconsistent bits of information, the use of a paraconsistent logic is the natural option. In particular, we do not reject classical logic: it has its own domains and applications. To this extent, while dealing with distinct domains, paraconsistent logic and classical logic are complementary rather than rivals. (They become rivals only when applied to the *same* domain. The rivalry derives from the fact that they provide different accounts of the logical connectives.)

But in applying paraconsistent logic, as we have done when formulating the theory of the Russell set and other inconsistent objects, we do not have to be committed to the existence of 'inconsistent entities' – this is the point of our claims (1) and (2) above. We can always use the resources provided by this logic only to help us draw consequences from inconsistent theories without triviality, but with no commitment to the *truth* of the theory in question; it can be at best partially true.

In this way, a non-committal (agnostic) interpretation of paraconsistency can be presented, making sense of the application of paraconsistent logic articulated here. 'Inconsistent objects', either mathematical or physical, can be accommodated without requiring an ontology which includes them. In particular, inconsistent mathematical

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[30] Descartes once remarked (in his *Principles*, iv, 204) that, 'with regard to those things that our senses cannot perceive, it suffices to explicate how they can be'. We would say the same with regard to inconsistent theories. The notion of partial truth allows us to accommodate formally this point, since partial truth is strictly weaker than truth, and does not commit us to anything *beyond* the assertion that certain structures are *possible*, given some paraconsistent logic.

theories can be studied in the context of paraconsistent logic, but it is not necessary to countenance the existence of the entities the theories are taken to be about.

8 Concluding remarks

We noted above that it is nearly impossible for anyone to follow all the literature on paraconsistent logics. We have left untouched numerous important topics here. Nowadays, ‘paraconsistency’ became a field of knowledge so wide that only events such as World Congresses on Paraconsistency can begin to do justice to it.

As a curiosity, a search for ‘paraconsistent logic’ on Google (`www.google.com`) on October 13, 2005 yielded 61,600 entries! We should stop now.

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