On the concept of identity in Zermelo-Fraenkel-like axioms and its relationship with quantum statistics

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Abstract

Georg Cantor said that a set is a collection into a whole of defined and *distinct* objects. This intuitive idea is in certain sense grasped by standard set theories, like Zermelo-Fraenkel's (ZF), where we can always (at least in principle) distinguish among the elements of a set. So, a natural question is: How to treat as 'sets' collections of *indistinguishable* objects, as those supposed by certain versions of quantum physics? Motivated by these issues, we have developed quasi-set theory. The problem of treating as 'sets' collections of entities like electrons, which would be indistinguishable, was listed as the first problem in list of Present Day Problems of Mathematics, posed at the Congress on the Hilbert Problems in 1974, sponsored by the American Mathematical Society. Embedded in such a context, quasi-set theory acquires a strong commitment to the way quantum physics copes with collections of elementary particles, once it admits particles of some sort. In this paper, we discuss the axioms of quasi-set theory and sketch some of its possible applications to physics. The discussion, we hope, will also help us to get a deeper understanding on the role played by the concept of identity in mathematics.

Keywords: Identity, Quasi-Sets, Axiomatic Set Theory, Indistinguishability, Quantum Statistics

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1 Introduction

It is well known that certain formulations of quantum theory deal with elementary 'particles' as entities which may be absolutely indistinguishable, in the sense of sharing all their physical properties. In classical physics, such a situation is quite different, for two particles are never allowed (by nature) to share the same state during the same time interval. Classical particles (the particles described by classical particle mechanics) are always distinguishable by either their intrinsic properties or by their state properties. But in quantum mechanics (once admitted particles) there may exist elementary particles which share the same intrinsic properties *and* the same quantum state. In this case, they are said to be indistinguishable ('identical' by the physicists). But, from the point of view of standard mathematics, indistinguishability entails identity. Hence, it would be impossible (in principle) to have a coherent mathematical picture of two or more elementary indistinguishable particles. If we aim at to define a mathematical framework for considering the possibility of having multiple collections of indistinguishable particles (indistinguishable with respect to all physical properties that describe a particle), then two ways seem to be open to cope with this problem: either we admit that there is some hidden property (besides intrinsic and state properties) that allows the existence of multiple objects with the same physical (not hidden) properties, or we should admit some new mathematic framework where indistinguishability does not entail identity. The first solution demands some sort of metaphysical hypothesis [23]. The second solution seems to demand new mathematical foundations for quantum mechanics. In the present paper we begin to explore the second alternative.

Physicists refer to particles that share the same physical properties as 'identical', while philosophers prefer to call them *indistinguishable*, since in the standard philosophical jargon, identical things are to be the very same thing. Standard set theories like Zermelo-Fraenkel, which sustain not only most of standard mathematics, but also most of theoretical physics, are so that sets (and their elements) obey a well defined theory of identity, according to which, roughly speaking, two objects a and b are always either identical (in the philosophical sense) or distinct (not identical). If a and b are distinct, then there exists a set c such that $a \in c$ bur $b \notin c$ (in extensional contexts, we might say that there exists at least one property -whose extension is c, which distinguishes them, a result which may be considered as a consequence of the so-called Leibniz Law -LL).[1] In particular, the elements of a set can

[1]Roughly speaking, in a second order language, if a and b are individual variables and F is a variable for properties of individuals, then LL can be written as $\forall F(F(a) \leftrightarrow F(b)) \leftrightarrow a = b$. Even in first order standard theories, like ZF, LL appears in a way, due to the axiom of extensionality and the postulates for identity of first order logic. Here, we shall

always be considered as *individuals* of a sort, since (at least ideally) they can be counted, ordered, and named.

In this sense, standard set theories cannot deal with collections of 'genuine' indistinguishable objects. The standard way mathematicians consider indistinguishable things vary, but all of them can, in some way or another, be summed up by a technique used by H. Weyl to treat aggregates of individuals [29, App. B]; in short, starting from a *set* S with, say, n elements, Weyl has assumed that there is an equivalence relation R defined on S, and then he take the equivalence classes $C_1, \ldots, C_k$ to play the role of collections of indistinguishable objects. So, what Weyl did was 'to forget' the very nature of S as a collection of distinct objects, and considered only the quantity of elements of S there are in each equivalence classes. As he says, this is what imports to physics. He might be right, but from the foundational point of view it seems that something is lacking, namely, *an adequate* treatment of indistinguishable objects as such just from the beginnings.

Other ways of treating the same question can be given by the introduction of invariance of some sort. For instance, we may consider as indistinguishable the elements which belong to the orbit of a certain element in a suitable group.[2] But all these 'solutions' are mathematical tricks, for the very characteristics of the elements of a set as *individuals* is always present, at least implicitly. So, this kind of devices cannot be considered as adequate answers to the *philosophical* problem of dealing with collections of indistinguishable objects.

But why this kind of problem, namely, that of dealing with collections of indistinguishable objects from the point of view of a 'set' theory, is so important? From a historical perspective, let us recall that this is precisely the first problem of the list of Problems of Present Day Mathematics, which appeared in the Congress on the Hilbert Problems, organized by the American Mathematical Society in 1974. The motivation for the stating of this problem is of course quantum physics, which (in some of its versions) deals with indistinguishable objects; as put by Yuri Manin,

"We should consider possibilities of developing a totally new lan-

term 'Leibniz Law' indistinctly anyone of these formulations. But see more on this below.

[2]Similar restrictions can be made to the other techniques usually considered by mathematicians, for instance the idea that indistinguishable objects are those elements which keep invariant a given structure under automorphisms (see, for example, [23], [16]).

guage to speak about infinity (...) I would like to point out that (...) [the usual language of set theory] is (...) an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the 'set' of grains of sand. In general, a highly probabilistic 'physical infinity' looks considerably more complicated and interesting than a plain infinity of 'things' (...) The twentieth century return to Middle Age scholastics taught us a lot about formalisms. Probably it is time to look outside again. Meaning is what really matters." [1, p. 36]

Of course the problem is not only to find a way of expressing indistinguishability. Physicists do this, for instance, by considering that only symmetric and antisymmetric vectors in an appropriate Hilbert space have a counterpart in the physical reality [22]. But let us insist once more that from the philosophical point of view, it should be interesting to consider indistinguishability *right at the start*, as something which is very peculiar to the objects being supposed to exist, as it seems to be the case, in some situations, with quantum objects [21]. In other words, if we take seriously the view that quantum objects shouldn't have individuality, that is, that they are to be taken as *non-individuals* in a sense (see [13]), can we present a 'set theory' where indistinguishability is introduced right from the start? Let us remark that the recourse of using permutation symmetries is a way of superseding the apparent impossibility of such language. The usual solution proposed by physicists by means of considering only symmetric and anti-symmetric states is just a description concerning states, nothing else. But quantum particles If they are to be supposed) are not quantum states, but are *associated* to quantum states. So, if we wish to talk about the particles themselves, then we should develop a mathematical solution to the problem of dealing with indistinguishable but not identical particles, that is, quasi-set theory.

We could also provide still another motivation for the development of quasi-set theory.[3] According to standard textbooks on statistical mechanics,

[3]The word 'objects' is used here as a neutral term, without any compromise with 'particles' or objects in the standard macroscopic sense; even waves or fields are can be 'objects' of a sort.

we know that Maxwell-Boltzmann (MB) 'statistics' gives us the most probable distribution of N *distinguishable* objects into, say, boxes with a specified number of objects in each box. In this case, we can show, e.g., that the hypothesis concerning distinguishable objects is unnecessary. Usually, classical and quantum distribution functions are mathematically derived in a naïve fashion; but an axiomatic framework is needed if we wish to show that individuality is not a necessary assumption in classical statistical mechanics. In a very interesting paper, N. Huggett [9] has shown that the occurrence of Maxwell-Boltzmann statistics in classical mechanics does not allow us to decide the metaphysical issue concerning molecules in a gas. In this paper, we also show that Maxwell-Boltzmann statistics is not committed to a metaphysical hypothesis concerning individuals.

2 Quasi-sets

In trying to develop a theory which deals with collections of indistinguishable objects in the sense mentioned in the previous section, we have taken the route of considering Erwin Schrödinger's idea that the concept of identity does not make sense for elementary particles. Of course this is not the only route to follow, but it is that one we shall pursue here, for interesting it is. In brief, this suggests that if x and y denote, say, electrons, it should be simply meaningless to say that x is identical (or different) from y [25, 26]. So, even taking this route, the resulting quasi-set theory provides a way of answering Manin's problem mentioned above. The theory presented here has some improvements if we compare it with previous versions given in [11] and [12].

The quasi-set theory $\mathcal{Q}$ is based on *ZFU*-like axioms (Zermelo-Fraenkel with *Urelemente*), but allows the existence of *two* sorts of atoms, termed m-atoms (also termed micro-atoms) and M-atoms (also termed macro-atoms). Two primitive unary predicates m and M help in expressing this idea: $m(x)$ says that x is an m-atom and $M(x)$ says that x is an M-atom, where x is a term. The language still encompasses the binary primitive predicates $\equiv$ (indistinguishability) and $\in$ (membership), one unary functional symbol qc (quasi-cardinal) and a unary predicate letter Z (where $Z(x)$ says that x is a *set*; these 'sets' are collections or quasi-sets that correspond precisely to the sets of *ZFU*). The basic idea is that the M-atoms shall have the properties of standard *Urelemente* of *ZFU*. Nevertheless, m-atoms do have a quite different behavior. Two indistinguishable micro-atoms are not necessarily identical (that is, $x \equiv y$ does not entail $x = y$). So, macro-atoms seem to be useful to describe the behavior of particles in classical particle mechanics (where all particles are distinguishable), while micro-atoms seem to be useful to describe whole collections of indistinguishable particles. Following Erwin Schrödinger, to this last kind of entities, we suppose that the concept of identity does not make sense, that is, $x = y$ is nor a well-formed formula if x and y denote m-atoms ([25, pp. 17-18]).

At this point it is worth to remark that when we talk about the 'traditional' concept of identity, we mean the theory of identity as presented in standard mathematics, either in first order or in higher order theories (and set theory) [19]. In quasi-set theory we make a restriction on the concept of formula: expressions like $x = y$ are not meaningful (well-formed formulas) if x and y denote m-atoms. The expression $x \equiv y$, which is read 'x is indistinguishable from y', and makes sense for all the objects we are considering. The equality symbol is not primitive in our theory, but a concept of *extensional identity* ($=_E$) is defined so that it has all the properties of standard identity of *ZFU*. Then, the axioms allow us to distinguish between the concepts of (extensional) *identity* (being the very same object) and *indistinguishability* (agreement with respect to all attributes), which cannot be done in classical logic and set theory.

A quasi-set (qset for short) x is defined as something which is not a *Urelement*. A qset x may have a cardinal (termed its *quasi-cardinal*, denoted by $qc(x)$) but, in general, it has not an ordinal, since there are quasi-sets which cannot be ordered (since their elements are indistinguishable m-atoms). The concept of quasi-cardinal is taken as primitive, since it cannot be defined by usual means. This fits the idea that quantum particles cannot be ordered or counted, but only aggregated in certain amounts. Notwithstanding, due to the concept of quasi-cardinal, there is a sense (as in orthodox quantum physics) in saying that there may exist a certain quantity of m-atoms obeying certain conditions, despite the fact that they cannot be named or labeled.

The primitive relation of indistinguishability ($\equiv$) is postulated to be reflexive, symmetric and transitive, but in order to make it different from identity (as ascribed by the traditional –first-order– theory of identity), the substitutivity axiom of equality does not hold in general, but only for some very specific cases. Even so, it should be interesting that such a relation of indistinguishability turns to be the standard identity (here represented by the extensional identity defined below) when the objects under consideration are not m-atoms. Then, the concept of *extensional identity* fits the idea of classical identity. The first definitions (nominal definitions) and axioms are the following:

Definition 1

1. *$Q(x) := \neg(m(x) \vee M(x))$. We read $Q(x)$ as "x is a quasi-set" or "x is a qset" for short.*

2. *$P(x) := Q(x) \wedge \forall y(y \in x \Rightarrow m(y)) \wedge \forall y \forall z(y \in x \wedge z \in x \Rightarrow y \equiv z)$. In this case we say that x is a* pure qset*.*

3. *$D(x) := M(x) \vee Z(x)$. These are the '(classical) things', to use Zermelo's original terminology. We read $D(x)$ as "x is a* Dinge*".*

4. *$E(x) := Q(x) \wedge \forall y(y \in x \Rightarrow Q(y))$.*

5. *$x =_E y := (Q(x) \wedge Q(y) \wedge \forall z(z \in x \Leftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge \forall_Q z(x \in z \Leftrightarrow y \in z))$. In this case we say that x and y are extensionally identical. The symbol "$=_E$" is called extensional identity.*

6. *$x \subseteq y := \forall z(z \in x \Rightarrow z \in y)$.*

The first item above says that a quasi-set is something which is not an atom, since it is neither a micro-atom nor a macro-atom. In other words, all terms are either atoms (micro or macro) or quasi-sets (which are collections of a kind). The second item says that a *pure* quasi-set is a quasi-set whose elements are all indistinguishable micro-atoms. The third item says that a *Dinge* is a term that behaves 'classically'. In other words, a *Dinge* behaves like a term in ZFU set theory, as shown in Corollary 1. According to the fourth item, if $E(x)$, then x is a quasi-set whose elements are quasi-sets. The fifth item says when two objects are extensionally identical. The last item is the standard definition of subset, but applied to quasi-sets.

In order to state the axioms of $\mathcal{Q}$ we need to remark that we use indexed quantifiers in order to abbreviate some formulas. If A is a given formula, P is a predicate letter and x is a variable, then the string $\forall_P x(A)$ means that $\forall x(P(x) \Rightarrow A)$. Analogously, $\exists_P x(A)$ means that $\exists x(P(x) \wedge A)$.

Here, we shall not make explicit the postulates of the underlying logic of $\mathcal{Q}$, which are similar to those of first order predicate calculus without identity. The specific axioms of $\mathcal{Q}$ are:

(Q1) $\forall x(x \equiv x)$

(Q2) $\forall x \forall y(x \equiv y \Rightarrow y \equiv x)$

(Q3) $\forall x \forall y \forall z(x \equiv y \wedge y \equiv z \Rightarrow x \equiv z)$

(Q4) $\forall x \forall y(x =_E y \Rightarrow (A(x,x) \Rightarrow A(x,y)))$, with the usual syntactic restrictions, i.e., $A(x,x)$ is a formula and $A(x,y)$ is obtained from $A(x,x)$ by replacing at least one of the free occurrences of x by y, if y is free for x in $A(x,x)$.

The first three axioms say that indistinguishability has the properties of an equivalence relation. The fourth axiom says that substitutivity can take place among terms that are extensionally identical. This means that $x =_E y$ entails $x \equiv y$, although the converse is not always valid.

Theorem 1 *Whether $Q(x)$ or $M(x)$, then $x =_E x$.*

Proof: If $Q(x)$, since $\forall z(z \in x \Leftrightarrow z \in x)$, then $x =_E x$ by the definition of extensional identity. If $M(x)$, then since $x \equiv x$ by **Q1**, it follows that $x =_E x$. ▮

Corollary 1 *The relation of extensional equality has all the properties of classical equality in first order theories.*

Proof: Straightforward, if we take into account the above theorem and **Q4**. In other words, extensional equality is something like a binary equivalence relation to which substitutivity holds. ▮

(Q5) Nothing is at the same time an m-atom and an M-atom:

$$\forall x(\neg(m(x) \wedge M(x)))$$

Theorem 2 *Wheter $Q(x)$ or $M(x)$, then $\neg m(x)$.*

Proof: If $Q(x)$, then $\neg m(x)$ by the definition of qset. If $M(x)$, then $\neg m(x)$ by **Q5**. ▮

(Q6) The atoms are empty:

$$\forall x \forall y (x \in y \Rightarrow Q(y))$$

(Q7) Every set is a qset:

$$\forall x (Z(x) \Rightarrow Q(x))$$

(Q8) Qsets whose elements are 'classical things' (*Dinge*) are sets and conversely:

$$\forall_Q x (\forall y (y \in x \Rightarrow D(y)) \Leftrightarrow Z(x))$$

What is the meaning of **Q8**? Our aim is to characterize *sets* in $\mathcal{Q}$ so that they can be identified with the sets of ZFU. This is supposed to be the case if they were taken to be those qsets whose transitive closure (this concept can be defined in the usual sense) does not contain m-atoms. The '$\Rightarrow$-part' of **Q8** gives half of the answer: if all the elements of x are *Dinge* (either sets or M-atoms), then x is a set. Concerning the converse, it is not enough to postulate that no element of a set is an m-atom, since it may be the case that the elements of its elements have m-atoms as elements and so on. The problem can be dealt with by taking $Z(x) \Rightarrow \forall y (y \in x \Rightarrow D(y))$, which is precisely the '$\Leftarrow$-part' of **Q8**.

(Q9)

$$\forall x (m(x) \wedge x \equiv y \Rightarrow m(y)) \wedge \forall x \forall y (x =_E y \wedge M(x) \Rightarrow M(y))$$

$$\wedge \forall x \forall y (x =_E y \wedge Z(x) \Rightarrow Z(y))$$

(Q10) There exists a qset denoted by '$\emptyset$' (the empty qset) , which does not have elements:

$$\exists_Q x \forall y (\neg (y \in x))$$

Theorem 3 *The empty qset is a set.*

Proof: Take $x =_E \emptyset$. Since $y \in x$ is false by **Q10**, then the antecedent of $\forall y (y \in x \Rightarrow D(x))$ is true, hence $Z(x)$ by **Q8**. ∎

(Q11) Indistinguishable *Dinge* are extensionally identical:

$$\forall_D x \forall_D y (x \equiv y \Rightarrow x =_E y)$$

(Q12) This is the qset-theoretical version of the weak-pair axiom. For all x and y, there exists a qset whose elements are indistinguishable from either x or y:

$$\forall x \forall y \exists_Q z \forall t (t \in z \Leftrightarrow t \equiv x \vee t \equiv y)$$

We denote this qset by $[x, y]$, and by $\{x, y\}$ when x and y are *Dinge*, according to standard terminology.

As we see below, after having discussed the idea of the quantity of elements of a qset (by means of the primitive concept of quasi-cardinal), the quasi-cardinal of $[x]$ may be different from 1, where $[x] := [x, x]$ is the 'weak singleton' of x.

(Q13) The Separation Schema: by considering the usual syntactical restrictions on the formula $A(t)$, the following is an axiom:

$$\forall_Q x \exists_Q y \forall t (t \in y \Leftrightarrow t \in x \wedge A(t))$$

This qset is written $[t \in x : A(t)]$ (we may use { and } when such a qset is a set).

(Q14) Union

$$\forall_Q x (E(x) \Rightarrow \exists_Q y (\forall z (z \in y) \Leftrightarrow \exists t (z \in t \wedge t \in x)))$$

This qset is denoted by $\bigcup_{t \in x} t$ (we also use $x \cup y$ as usual).

(Q15) Power-qset

$$\forall_Q x \exists_Q y \forall t (t \in y \Leftrightarrow t \subseteq x)$$

According to the standard notation, we write $\mathcal{P}(x)$ for this qset.

Definition 2

1. $\langle x, y \rangle := [[x], [x, y]]$

2. $x \times y := [\langle z, u\rangle \in \mathcal{PP}(x \cup y) : z \in x \wedge u \in y]$

3. *The concepts of intersection and difference of qsets are defined in the usual way so that* $t \in x \cap y$ *iff* $t \in x \wedge t \in y$ *and* $t \in x - y$ *iff* $t \in x \wedge t \notin y$. *This last concept will be mentioned again later. It is worth to note that the symbol '*$\notin$*' has its usual meaning as in set theory.*

We remark that $\langle x, y\rangle$ is a kind of 'generalized ordered pair', since the first element is the qset of all elements indistinguishable from x, while the second is the qset of all elements indistinguishable from y. We call it the 'weak pair'.

(Q16) Infinity:

$$\exists_Q x(\emptyset \in x \wedge \forall y(y \in x \wedge Q(y) \Rightarrow y \cup [y] \in x))$$

.

(Q17) Regularity: (Qsets are well-founded):

$$\forall_Q x(E(x) \wedge x \neq_E \emptyset \Rightarrow \exists_Q y(y \in x \wedge y \cap x =_E \emptyset))$$

2.1 Relations

Definition 3 *A qset w is a relation if it satisfies the following predicate R:*

$$R(w) := Q(w) \wedge \forall z(z \in w \Rightarrow \exists u \exists v(u \in x \wedge v \in y \wedge z =_E \langle u, v\rangle))$$

Theorem 4 *No partial, total or strict order relation can be defined on a pure qset whose elements are indistinguishable from one another.*

Proof: (Sketch) Partial and total orders require antisymmetry, and this property cannot be stated without identity. Asymmetry also cannot be supposed, for, if $x \equiv y$, then for every R such that $\langle x, y\rangle \in R$, and there it follows that $\langle x, y\rangle =_E [[x]] =_E \langle y, x\rangle \in R$; so, xRy entails yRx. ∎

Theorem 5 *There exists a translation from the language of ZFU into the language of $\mathcal{Q}$ such that if A is a formula of* ZFU *and A^q is its translation to the language of $\mathcal{Q}$, then $\vdash_{ZFU} A$ iff $\vdash_{\mathcal{Q}} A^q$. In other words, there is a translation from one language into the other one such that all translations of theorems of ZFU are theorems of $\mathcal{Q}$.*

Proof: The theory $\mathcal{Q}$ encompasses a 'classical' counterpart which can be defined as follows: let A be a formula of the language of *ZFU* (which we may admit that has an unary predicate S which stands for 'sets'). Then, call A^q its translation to $\mathcal{Q}$, defined as follows, where $S(x)$ means that x is a set (in ZFU):

1. If A is $S(x)$, then A^q is $Z(x)$
2. If A is $x = y$, then A^q is $((M(x) \wedge M(y)) \vee (Z(y) \wedge Z(y)) \wedge x =_E y)$
3. If A is $x \in y$, then A^q is $((M(x) \vee Z(x)) \wedge Z(y)) \wedge x \in y$
4. If A is $\neg B$, then A^q is $\neg B^q$
5. If A is $B \vee C$, then A^q is $B^q \vee C^q$
6. If A is $\forall x B$, then A^q is $\forall x(M(x) \vee Z(x) \Rightarrow B)$

Then it is easy to see that the translations of the axioms of *ZFU* are theorems of $\mathcal{Q}$. So, if $\mathcal{Q}$ is consistent, so is *ZFU* (see [3]). ▮

The above theorem shows that there is a *copy* of *ZFU* in $\mathcal{Q}$. The existence of such a copy of ZFU tells us that standard mathematics is a particular case of quasi-set-theoretical mathematics. So, we can still use all the standard results of classical mathematics (grounded on ZFU) in the quasi-set-theoretical framework. In this 'copy' of ZFU, we may define the following concepts: $Cd(x)$ for 'x is a cardinal'; $card(x)$ denotes 'the cardinal of x', and $Fin(x)$ says that 'x is a finite quasi-set' (that is, $qc(x)$ is a natural number, these ones defined as usual in the 'classical part' of the theory).

In other words, the concept of quasi-cardinal one of the primitive notions of $\mathcal{Q}$), but the concept of *cardinal* is definable in the usual way, since ZFU is copied in quasi-set theory. The details (which are straightforward) are left to the reader.

By considering these concepts, we may present the axioms for quasi-cardinals:

(Q18) Every object which is not a qset (that is, every *Urelement*) has quasi-cardinal zero:

$$\forall x(\neg Q(x) \Rightarrow qc(x) =_E 0)$$

(Q19) The quasi-cardinal of a qset is a cardinal (defined in the 'classical part' of the theory and coincides with the cardinal itself when this qset is a set:

$$\forall_Q x \exists! y(Cd(y) \wedge y =_E qc(x) \wedge (Z(x) \Rightarrow y =_E card(x)))$$

(Q20) Every non-empty qset has a non zero quasi-cardinal:

$$\forall_Q x(x \neq_E \emptyset \Rightarrow qc(x) \neq_E 0)$$

(Q21) $\forall_Q x(qc(x) =_E \alpha \Rightarrow \forall \beta(\beta \leq_E \alpha \Rightarrow \exists_Q y(y \subseteq x \wedge qc(y) =_E \beta))$
(Q22) $\forall_Q x \forall_Q y \forall t(y \subseteq x \rightarrow qc(y) \leq_E qc(x))$
(Q23) $\forall_Q x \forall_Q y(Fin(x) \wedge x \subset y \Rightarrow qc(x) < qc(y))$
(Q24) $\forall_Q x \forall_Q y(\forall w(w \notin x \vee w \notin y) \Rightarrow qc(x \cup y) =_E qc(x) + qc(y))$

In the next axiom, $2^{qc(x)}$ denotes (intuitively) the quantity of subquasi-sets of x. Then,
(Q25) $\forall_Q x(qc(\mathcal{P}(x)) =_E 2^{qc(x)})$

Axiom **Q25** raises an interesting discussion that we mention below. But first we need the concept of *quasi-function*.

2.2 Quasi-functions

Standard functions could not distinguish between arguments and values. So, we have:

Definition 4 *If x and y are qsets and R is the predicate for 'relation' defined above, we say that f is a* quasi-function *(qfunction) if it satisfies the following predicate:*

$$QF(f) := R(f) \wedge \forall u(u \in x \Rightarrow \exists v(v \in y \wedge \langle u, v\rangle \in f))\wedge$$

$$\forall u \forall u' \forall v \forall v'(\langle u, v\rangle \in f \wedge \langle u', v'\rangle \in f \wedge u \equiv u' \Rightarrow v \equiv v')$$

f is a q-injection if f is a q-function from x to y and satisfies the additional condition:

$\forall u \forall u' \forall v \forall v'(\langle u, v\rangle \in f \wedge \langle u', v'\rangle \in f \wedge v \equiv v' \Rightarrow u \equiv u') \wedge qc(Dom(f)) \leq_E qc(Rang(f))$

f is a q-surjection if it is a function from x to y such that

$\forall v(v \in y \Rightarrow \exists u(u \in x \wedge \langle u, v\rangle \in f)) \wedge qc(Dem(f)) \geq_E qc(Rang(f))$.

A function f which is both a q-injection and a q-surjection is said to be a q-bijection. In this case, $qc(Dom(f)) =_E qc(Rang(f))$.

2.3 How many sub-quasi-sets there are?

Now we can turn to the discussion involving axiom **Q25**. Since the concept of identity has no meaning for m-atoms, how can we ensure that a qset x such that $qc(x) =_E \alpha$, has precisely 2^α subqsets? In standard set theories (and in the 'classical part' of $\mathcal{Q}$, that is, those qsets which copy the sets of ZFU), as it is well known, if $card(x)$ denotes the cardinal of x, then by the definition of exponentiation of cardinals, $2^{card(x)}$ is defined to be the cardinal of the set ${}^x2$, which is the set of all functions from x to the Boolean algebra $2 = \{0, 1\}$ (see [4]). In $\mathcal{Q}$ this definition does not work. Let us explain why.

Suppose that α is the quasi-cardinal of x, which is a cardinal, by axiom **Q19**. This axiom says that every qset has a unique quasi-cardinal which is a cardinal (defined in the 'classical part' of the theory), and if the qset is in particular a set (in $\mathcal{Q}$), then this quasi-cardinal is its cardinal *stricto sensu*. So, every quasi-cardinal is a cardinal and the expression 'there is a unique' makes sense. Furthermore, from the fact that $\emptyset$ is a set, it follows that its quasi-cardinal is 0. Then we may write

$$2^{qc(x)} := qc({}^\alpha 2) \quad (1)$$

and then, since α is a cardinal and both α and 2 are *qsets*, we have

$$2^{qc(x)} := card({}^\alpha 2) \quad (2)$$

So, we may take the cardinal of the qset ${}^\alpha 2$ in its usual sense to mean $2^{qc(x)}$. Then, equation (2) entails a meaning to axiom **Q25**, since it explains what does $2^{qc(x)}$ mean: it is the cardinal of the *set* of all the applications from α (the quasi-cardinal of x) to 2. By considering this, the axiom may be written as follows, where x is a qset and α is its quasi-cardinal:

Axiom Q25 (Alternative Form)

$$\forall_Q x(qc(\mathcal{P}(x))) =_E card({}^\alpha 2).$$

We remark that the second member of the equality has a precise meaning in $\mathcal{Q}$, since both α and 2 act as in classical set theories, as remarked above. This characterization allows us to avoid another problem, which could be thought to be derived in quasi-set theory. In order to explain this we recall that in standard set theories we can prove that $\mathcal{P}(x)$ is equinumerous with ${}^{x}2$ by defining a one-to-one function $f : \mathcal{P}(x) \rightarrow^{x} 2$ as it follows: for every $y \subseteq x$, let $f(y)$ be the characteristic function of y, namely, the function $\chi_y : x \rightarrow 2$ defined by

$$\chi_y(t) := \begin{cases} 1 & \text{if } t \in y \\ 0 & \text{if } t \in x - y \end{cases} \tag{3}$$

Then any function $h \in {}^{x}2$ belongs to the range of f since

$$h = f(\{t \in x : h(t) = 1\}).$$

Suppose now that x is a qset such that $qc(x)$ is the natural number n and that all elements of x are indistinguishable (the natural numbers are defined in $\mathcal{Q}$ in the usual way, in the copy of *ZFU* that we have defined in $\mathcal{Q}$). For all we need, it is enough to consider finite qsets (this definition is also standard by taking the concept of function given above). In this case, we cannot define the characteristic quasi-function χ_y^q for $y \subseteq x$, since, for instance, if $\chi_y^q(t) =_E 1$ for $t \in y$, then $\chi_y^q(w) =_E 1$ as well as for every $w \in x$, independently if either w belongs to y or not. This is due to the definition of quasi-functions given above, since for every quasi-function f,

$$\langle a, b \rangle \in f \wedge \langle c, d \rangle \in f \wedge a \equiv c \Rightarrow b \equiv d.$$

In other words, if the image of a certain t by the quasi-function f is 1, then the image of every element that is indistinguishable from t will be 1 as well. So, $\mathcal{Q}$ distinguishes only between *two* quasi-functions from x to 2, namely, that one which associates 1 to all elements of x and that one which associates 0 to all of them. This is why we have used $qc({}^{\alpha}2)$ to stand for $2^{qc(x)}$, since both α and 2 may be viewed as *sets* (in the standard sense of ZF). If we had used ${}^{x}2$ instead, we would be unable to distinguish among certain quasi-functions, so complicating the meaning of **Q25**, since we could have no manner of counting the number of sub-quasi-sets of a qset. But, by using ${}^{\alpha}2$, since both α and 2 behave 'classically', we may keep **Q25** with its usual meaning.

From these considerations, we may conclude that when x is a qset whose elements are indistinguishable m-atoms, we cannot prove *within* $\mathcal{Q}$ that if $qc(x) =_E n$, it is not possible to assert that x has 2^n sub-quasi-sets. Since this is precisely what **Q25** intuitively means, we may say that this axiom cannot be proven from the remaining axioms of $\mathcal{Q}$. But, since it holds for particular qsets, namely, to those which are *sets*, it cannot be disproved as well. In order to state that **Q25** cannot be disproved, consider the *sets* in $\mathcal{Q}$; since they behave as classical sets, we can prove that what **Q25** asserts is true. Now it suffices to take a qset whose elements are indistinguishable m-atoms and such that $qc(x) = \alpha$.

2.4 The 'weak' extensionality

The absence of a theory of identity for m-atoms causes the necessity of a modification in the Axiom of Extensionality, which does not hold here as in standard set theories. In order to do so, let us introduce the following definition:

Definition 5 *For all non empty quasi-sets x and y,*
$Sim(x, y) := \forall z \forall t(z \in x \wedge t \in y \Rightarrow z \equiv t)$ *In this case we say that x and y are* similar.

$QSim(x, y) := Sim(x, y) \wedge qcard(x) =_E qcard(y)$*. That is, x and y are q-*similar *iff they are similar and have the same quasi-cardinality.*

(Q26) Weak Extensionality: Qsets which have the same quantity of elements of the same sort are indistinguishable. In symbols,

$$\forall_Q x \forall_Q y((\forall z(z \in x/\equiv \Rightarrow \exists t(t \in y/\equiv \wedge \wedge QSim(z, t))))$$

$$\wedge \forall t(t \in y/\equiv \Rightarrow \exists z(z \in x/\equiv \wedge \wedge QSim(t, z))) \Rightarrow x \equiv y)$$

Axiom **Q26** allows us to make another remark about **Q25**. As in standard set theories, if $card(x) =_E n$, then are there exactly n subqsets of x which are singletons? If not, how can we make sense to the idea that if $qc(x) =_E n$, then x has n elements? We recall that the main motivation of $\mathcal{Q}$ is the way quantum mechanics deals with elementary particles. In this theory, although there is a sense in saying that, say, there are k electrons in a certain level of a certain atom, there is no way of counting them or distinguishing among them (see [27, Chap. 12]).

If x is a qset whose elements are indistinguishable from one another (let us suppose again that $qc(x) =_E n$, which suffices for our purposes), then the singletons $y \subseteq x$ are indistinguishable, as results from the weak extensionality axiom **Q26**. So, all the singletons (in the intuitive sense) seem to fall in just one qset. But it should be recalled that these 'singletons' (subqsets whose quasi-cardinality is 1) are not *identical* (that is, they are not *the same object*), but they are indistinguishable in the sense given by **Q26**. In other words, despite the theory cannot distinguish among them, we cannot state neither that they are the same qsets nor that their elements are identical. So, it is consistent with $\mathcal{Q}$ to suppose that if $qc(x) = \alpha$, then x has precisely α 'singletons'. So, due to Q25, the theory does not forbid the existence of such singletons, despite the fact that in $\mathcal{Q}$ we cannot prove that they exist as 'distinct' entities, and hence we may reason in $\mathcal{Q}$ as physicists do when informally dealing with a certain number of indistinguishable elementary particles.

By means of **Q26** it is easy to prove the following theorem:

Theorem 6

1. $x =_E \emptyset \wedge y =_E \emptyset \Rightarrow x \equiv y$
2. $\forall_Q x \forall_Q y (Sim(x, y) \wedge qc(x) =_E qc(y) \Rightarrow x \equiv y)$
3. $\forall_Q x \forall_Q y (\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x \equiv y)$
4. $x \equiv y \wedge qc([x]) =_E qc([y]) \Leftrightarrow [x] \equiv [y]$

2.5 Replacement axioms

To keep $\mathcal{Q}$ with a structure similar to ZFU, we might state Replacement Axioms as follows, whose consequences are not explored here:

If $A(x, y)$ is a formula in which x and y are free variables, we say that $A(x, y)$ defines a $y-(qfunctional)$ condition on the quasi-set t if $\forall w(w \in t \Rightarrow \exists s A(w, s) \wedge \forall w \forall w'(w \in t \wedge w' \in t \Rightarrow \forall s \forall s'(A(w, s) \wedge A(w', s') \wedge w \equiv w' \Rightarrow s \equiv s'))$ (this is abbreviated by $\forall x \exists! y A(x, y)$). Then, we have:

(Q27) Replacement
$\forall x \exists ! y A(x,y) \Rightarrow \forall_Q u \exists_Q v (\forall z (z \in v \Rightarrow \exists w (w \in u \wedge A(w,z)))$

2.6 The concept of strong singleton

Definition 6 *A* strong singleton *of x is a quasi-set x' which satisfies the following property:*

$$x' \subseteq [x] \wedge qc(x') =_E 1$$

In words, a strong singleton of x is a qset whose only element is indistinguishable from x. In standard set theories, this qset is of course the singleton whose only element is x itself, but here x may be an m-atom, and in this case there is no way of speaking of something *being identical to x*. Even so, we can prove that such a qset exists:

Theorem 7 *For all x, there exists a strong singleton of x.*

Proof: The qset $[x]$ exists according to the weak pair axiom. Since $x \in [x]$ (recall that $\equiv$ is reflexive), we have $qc([x]) \geq_E 1$ by **Q20**. But, from **Q21**, there exists a subqset of $[x]$ which has quasi-cardinal 1. Take this qset to be x'. ▮

Theorem 8 *All the strong singletons of x are indistinguishable.*

Proof: Immediate consequence of **Q26**, since all of them have the same quasi-cardinality 1 and their elements are indistinguishable by definition. ▮

We remark that, as we shall see, we cannot prove that the strong singletons of x are extensionally identical. Intuitively, a strong singleton of x is a qset of quasi-cardinality 1 whose 'only element' is indistinguishable from x. But even if $y \equiv x$, we cannot get as a theorem of $\mathcal{Q}$ that the x and y are *the same object*, for to express this we need identity.

The concept of difference of qsets is introduced in the usual way: $x - y$ is the qset whose elements are the elements of x which do not belong to y. Intuitively, the we can think of the electrons of a certain level inside an atom which are not in another level, although we have no practical means to select an electron and say: 'that's the electron we are talking about'. The theory express formally such idea; in other words, concerning indistinguishable m-atoms, we cannot give ostensive definitions, say, by pointing one finger over an m-atom and saying 'That is Peter'. Even so, like in quantum physics, we may reason in $\mathcal{Q}$ as if a certain element either belongs to the qset or does not; the excluded middle law remains valid, even if we cannot verify what case holds (this is a kind of non-constructive reasoning). Insisting a little bit, we recall that also in standard mathematics there is no general effective way of proving that, given inputs x and y, if $x \in y$ or $x \notin y$, although one of them is true (since the excluded middle law holds in classical logic). This idea fits what happens with the electrons in an atom; in general we know how many electrons there are in certain situations, say in a specific atom, and we can say that some of them *are* in that atom, but we cannot identify them: to ask which are the electrons is a meaningless question. To speak of something being meaningless deserves some care. But perhaps we are in a situation which resembles Heisenberg, when he explains why, from the point of view of modern physics, the problem posed by the ancient atomists of looking for the ultimate parts of matter also 'has no meaning':

> We ask, 'What does the proton consist of?' 'Is the light-quantum simple, or is it composite?' But these questions are wrongly put, since the words *divide* or *consist of* have largely lost their meaning. It would thus be our task to adapt our language and thought, and hence also our scientific philosophy, to this new situation engendered by the experiments. ([8, p. 82])

Coming back to $x - y$, we will show that the quasi-cardinal of $x - y$ is, as expected, $qc(x) - qc(y)$.

Theorem 9 *For all qsets x and y, if $y \subseteq x$, then $qc(x-y) =_E qc(x) - qc(y)$.*

Proof: By definition, $t \in x - y$ iff $t \in x \wedge t \notin y$. Then $(x - y) \cap y =_E \emptyset$. Hence, by **Q24**, $qc((x - y) \cup y) =_E qc(x - y) + qc(y)$ (let us call this expression (i)). But, since $y \subseteq x$, $(x - y) \cup y =_E x$ and so, in order that (i) be true, $qc(x - y) =_E qc(x) - qc(y)$.▮

The next result may be viewed as a quasi-set-theoretical version of the Indistinguishability Postulate used in quantum physics. Roughly speaking, it says that permutations of indistinguishable quanta are not observable, and constitute one of the most basic metaphysical assumptions which underlies quantum mechanics [13]. In order to state and prove this result, we need the following definition:

Definition 7 *1. Let x be a qset such that $E(x)$, that is (according to Definition 1), its elements are also qsets. Then,*

$$\bigcap_{t \in x} t := [z \in \bigcup_{t \in x} t : \forall s(s \in x \Rightarrow z \in s)]$$

2. If $m(u)$, then $S_u := [s \in \mathcal{P}([u]) : u \in s]$

3. $u^* := \bigcap_{t \in S_u} t$

Lemma 1 *If $m(u)$, then:*

1. $u \in \bigcup_{t \in S_u} t$

2. $\forall s(s \in S_u \Rightarrow u \in s)$

3. $z \in u^*$ iff $z \in \bigcup_{t \in S_u} t \wedge \forall s(s \in S_u \Rightarrow z \in s)$

4. $u \in u^*$

5. $u^* \subseteq [u]$

6. If $s \in S_u$, then $u^* \subseteq s$

Proof: (1) $z \in \bigcup_{t \in S_u} t$ iff $\exists t(t \in S_u \wedge z \in t)$. Therefore, from the definition given above, $z \in \bigcup_{t \in S_u} t$ iff $\exists t(t \in \mathcal{P}([u]) \wedge u \in t \wedge z \in t)$. But since $[u] \in \mathcal{P}([u])$ and $u \in [u]$, it follows that $u \in \bigcup_{t \in S_u} t$. (2) $\forall s(s \in S_u \Leftrightarrow s \in \mathcal{P}([u]) \wedge u \in s)$. Therefore, $\forall s(s \in S_u \Rightarrow u \in s)$. (c) Immediate consequence of the above definition. (4) Immediate consequence of (1)-(3) above. (5) Suppose that $z \in u^*$. By (3), we have $\forall s(s \in S_u \Rightarrow z \in s)$. But since $[u] \in S_u$, it results that $z \in [u]$. (6) If $z \in u^*$, then, as before, $\forall s(s \in S_u \Rightarrow z \in s)$. But, by hypothesis, $s \in S_u$; so, $z \in s$. ∎

Lemma 2 *If u is an m-atom and z is a qset, then if $z \subseteq u^*$ and $qc(z) =_E 1$, it results that either $u \in u^* - z$ or $qc(u^*) =_E 1$.*

Proof: Suppose that $u \notin u^* - z$. Since $u \in u^*$, it follows that $u \in z$. But $z \subseteq u^* \subseteq [u]$, therefore $z \in S_u$. But, by item (6) of the Lemma given above, $u^* \subseteq z$. By hypothesis, $z \subseteq u^*$, hence $u^* =_E z$, and so $qc(u^*) =_E qc(z) =_E 1$. ∎

Theorem 10 *For every u, $qc(u^*) =_E 1$.*

Proof: According to item (4) of Lemma (1), $u^* \neq_E \emptyset$. So, by **Q20**, $qc(u^*) \neq_E 0$, hence $qc(u^*) \geq_E 1$. We shall show that the equality holds. Suppose that $qc(u^*) >_E 1$. Then, by **Q21**, there exists a qset $w \subseteq u^*$ such that $qc(w) =_E 1$. So, by Lemma (2), $u \in u^* - w$. But $u^* - w \subseteq [u]$, since $u^* \subseteq [u]$. Therefore, $u^* - w \in S_u$. By Lemma (1), item (6), $u^* \subseteq u^* - w$. But since $u^* - w \subseteq u^*$, it follows that $u^* =_E u^* - w$. Again by **Q20**, $w \neq_E \emptyset$ since $qc(w) =_E 1$. Then let be $t \in w$. So, $t \in u^*$ since $w \subseteq u^*$, hence $t \in u^* - w$ (once $u^* =_E u^* - w$). Then $t \notin w$, a contradiction. ∎

Lemma 3 *For all m-atoms u and v, if $u \equiv v$, then $u^* \equiv v^*$. Furthermore, if $u \in w$, then $u^* \subseteq w$ for any qset w.*

Proof: By Lemma (1), item (5), $u^* \subseteq [u]$ and $v^* \subseteq [v]$; if $u \equiv v$ then $Sim(u^*, v^*)$ (see Definition (5)). But, by Theorem (10), $qc(u^*) =_E 1$ and $qc(v^*) =_E 1$ and then, by theorem (6), item (2), $u^* \equiv v^*$. The last part can be proven by noting that if $u \in w$, then $u \in w \cap [u]$, so as $w \cap [u] \subseteq [u]$, therefore $w \cap [u] \in S_u$. Then, by Lemma (1), item (6), $u^* \subseteq w \cap [u]$ and so $u^* \subseteq w$. ∎

These last results show that u^* is, as expected, one of the strong singletons of u. The remarkable fact, as already mentioned earlier, is that we cannot prove that $u^* \equiv v^*$ entails $u^* =_E v^*$. This is due to the fact that nothing in the theory can assure that *that* m-atom that belongs to u^* *is the same* m-atom that belongs to v^*, since neither the expression $u = v$ nor $u =_E v$ are well-formed formulas. Furthermore, it is interesting to recall that the usual Extensionality Axiom, which could be used for expressing this fact, is not an axiom of our theory but, instead, we have the 'weak' axiom **Q26**, which talks about indistinguishability only, but not about identity. The impossibility of proving the mentioned result should not be regarded as a deficiency of the theory, but rather as expressing that it is closer to what happens in some physical domains. We shall be back to this point below.

2.7 Permutations are not observable

The next theorem states within the language of $\mathcal{Q}$ the intuitive idea mentioned above, namely, that *permutations are not observable*. Let us recall that in standard set theories, if $z \in x$, then $(x - \{z\}) \cup \{w\} = x$ iff $z = w$. So, we shall prove the following theorem:

Theorem 11 *Let x be a qset such that $x \neq_E [z]$ and z an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists w' such that*

$$(x - z') \cup w' \equiv x$$

Proof: Case 1: $t \in z'$ does not belong to x. In this case, $x - z' =_E x$ and so we may admit the existence of w' such that its unique element s does belong to x (for instance, s may be z itself); then $(x - z') \cup w' =_E x$.

Case 2: $t \in z'$ does belong to x. Then $qc(x - z') =_E qc(x) - 1$ by the above Theorem. Then we take w' such that its element is w itself, and so it results that $(x-z') \cap w' =_E \emptyset$. Hence, by **Q25**, $qc((x-z') \cup w') =_E qc(x)$. This intuitively says that both $(x-z') \cup w'$ and x have the same quantity of indistinguishable elements. So, using **Q27** (see above), we obtain the theorem. ▮

When $w \notin x$, we have the desired case according to which the theorem is intuitively saying that we have 'exchanged' an element of x by an indistinguishable one, and that the resulting fact is that 'nothing has occurred at all', a fact that in quantum physics was nicely expressed by Roger Penrose when he said that

> "according to the modern theory [QM], if a particle of a person's body were exchanged with a similar particle in one of the bricks of this house then nothing would have happened at all". ([20, p. 360])

In other words, the resulting qset (after the permutation) is indistinguishable from the original one. The above theorem is the quasi-set-theoretical version of the quantum mechanical fact which expresses that permutations of indistinguishable particles are not regarded as an observable, as expressed by the so-called Indistinguishability Postulate. The relations between quasi-sets and quantum objects are discussed from different points of view in [13, 7, 15].

2.8 The Axiom of Weak Choice

Finally, we can add to the theory $\mathcal{Q}$, for instance, the following axiom of 'weak' choice.

(Q28) The Axiom of Weak Choice

$$\forall_Q x(E(x) \wedge \forall y \forall z(y \in x \wedge z \in x \Rightarrow y \cap z =_E \emptyset \wedge y \neq_E \emptyset) \Rightarrow$$
$$\exists_Q u \forall y \forall v(y \in x \wedge v \in y \Rightarrow \exists_Q w(w \subseteq [v] \wedge qc(w) =_E 1 \wedge w \cap y \equiv w \cap u)))$$

Of course this axiom is formulated only to keep $\mathcal{Q}$ strong enough to be compared with standard ZFU, as we have done also with the Replacement Axioms. In the axiom, the "choice qset" is formed by taking one indistinguishable from each member of the qset x. That is, we 'weakly' take *one* element from each sub-qset of x, but without naming it. This procedure resembles the use of Hilbert's ϵ-symbol, and we leave as open the problem of further investigating this relationship, including the fact that, as it is well known, Hilbert's symbol keeps the axiom of choice a theorem of standard set theory. It is also worth noting that since we can suppose the existence of qsets with quasi-cardinal 2 whose elements are indistinguishable m-atoms, we may also ask whether these qsets might act also as Fraenkel's "cells" [5] in order to obtain, as he did, a proof of the independence of the negation of the axiom of choice from the remaining axioms of $\mathcal{Q}$. As it is well known, the *Urelemente* of ZFU set theory are indistinguishable in the sense of being invariant under automorphisms, but even so they do obey the classical theory of identity (so they are *individuals*). Our m-atoms, instead, act as 'legitimate' indistinguishable entities. To philosophically pursue these questions might be interesting questions we leave to other works.

3 A closer look on identity

In this section we make some critical analysis on the foundations of equality (identity) in first order theories from the point of view of quasi-sets. The point of course concerns the meaning of the concept of indistinguishability in $\mathcal{Q}$, which is to be made distinct from identity. In considering this, we are led to the study of alternative formulations of quasi-set theory. Let us begin by recalling the axiom **Q4**, namely:

$$\forall x \forall y (x =_E y \Rightarrow (A(x,x) \Rightarrow A(x,y))).$$

The question is: what would happen if we rephrase this sentence in a somehow stronger way? One possibility is to replace $x =_E y$ by $x \equiv y$, which could be a natural supposition. So,

$$\forall x \forall y (x \equiv y \Rightarrow (A(x,x) \Rightarrow A(x,y))), \text{ with the usual restrictions.}$$

But, in this case, we can easily see that indistinguishability collapses to identity, that is, quasi-set theory reduces to ZFU. So, we should ask whether there is any another possible alternative for **Q4**, stronger than our **Q4**, but such that indistinguishability does not collapse to identity. We shall argue that there is not such a possibility and, in order to show that, we illustrate our ideas by means of an alternative version for **Q4**, which we call **Q4#**, where the usual restrictions are obeyed:

Q4# - $\forall x \forall y (\neg m(x) \wedge \neg m(y) \wedge x \equiv y \Rightarrow (A(x,x) \Rightarrow A(x,y)))$.

We can see that **Q4#** is stronger than **Q4** since **Q4#** allows substitutivity for indistinguishable qsets which are not extensionally identical. But the new question is: taken the axioms **Q1**- **Q3** plus **Q4#**, is the resulting concept of indistinguishability weaker than identity?

In order to discuss such a claim, we will prove some lemmas and theorems within the scope of a variant of our quasi set theory obtained by replacing **Q4#** for **Q4**. We call this theory $\mathcal{Q}\#$. Our main goal with this discussion is to arrive at a better understanding of identity in first order set theories. In short, we will prove that $\mathcal{Q}\#$ is equivalent to ZFU set theory, since indistinguishability, in this case, collapses to identity. In $\mathcal{Q}\#$, it is easy to prove the following lemmas:

Lemma 4 *For all qsets x and y we have:*

1. *If $t \in x$ and $x \equiv y$ then $t \in y$;*
2. *If $x \equiv y$ then $x =_E y$.*

Proof: (1) Since x is a qset, then $\neg m(x)$, according to definition (1). Since $x \equiv y$ then $\neg m(y)$ (**Q9**). Now, for all qsets u and v let $A_t(u, v) := (t \in u \wedge t \in v)$. By hypothesis, $t \in x \wedge t \in x$, i.e., $A_t(x, x)$. Since $x \equiv y$ then, according to **Q4#**, $A_t(x, y)$. Hence $t \in y$. (2) If $x \equiv y$ then, according to item 1 of this proof, $t \in x$ iff $t \in y$, which means that $x =_E y$. ∎

Lemma 5 *For all m-atom x,* $[x] =_E x^*$*, where* x^* *is given by definition (7).*

Proof: Let $t \in [x]$. According to the definition of weak singleton, $t \equiv x$. According to **Q9**, t is an m-atom. From lemma (3), we have $t^* \equiv x^*$. Note that the proof of lemma (3) does not make any reference to **Q4**, so we can use it here, although we are working in $\mathcal{Q}\#$. From definition (7) we know that t^* and x^* are qsets. So, $t^* =_E x^*$, according to lemma (4). So, since $t \in t^*$, then $t \in x^*$. Therefore $[x] \subseteq x^*$. From lemma (1) (which also does not make any reference to **Q4** in its proof) $x^* \subseteq [x]$. So, $[x] =_E x^*$.∎

Lemma 6 *For all m-atoms x and y, the following conditions are equivalent:*

1. $x \equiv y$;
2. $x^* \equiv y^*$;
3. $x^* =_E y^*$;
4. $[x] =_E [y]$;
5. $[x] \equiv [y]$.

Proof: By lemma (3) (1)⇒ (2); By **Q4#** (2)⇒(3); By lema (5) (3)⇒ (4); by theorem (6), item (3), (4)⇒ (5); By the same theorem, item (4), (5)⇒ (1).∎

Theorem 12 *For all m-atom x,* $qc([x]) =_E 1$*.*

Proof: Straightforward from lemma (5) and theorem (10) (which makes no reference to **Q4** in its proof).∎

Lemma 7 *For all m atoms x and y and for all qset w, if* $x \equiv y$ *and* $x \in w$*, then* $y \in w$*.*

Proof: From lemma (3) $x^* \subseteq w$. From lemma (6) $y^* \subseteq w$. From lemma (1), item (4), $y \in w$.▮

Lemma 8 *For all m-atoms x and y and for all qset w:*

1. *If $x \equiv y$ and $[x] \in w$, then $[y] \in w$;*
2. *If $x \equiv y$ and $[x] \subseteq w$ then $[y] \subseteq w$.*

Proof: If $x \equiv y$ then, from lemma (6), $[x] \equiv [y]$. But $[x]$ and $[y]$ are qsets. Then, from **Q4#** we have (1) and (2).▮

Lemma 9 *For all m-atoms x and y and for all λ and z:*

1. *If $x \equiv y$ and $qc([x]) =_E \lambda$ then $qc([y]) =_E \lambda$;*
2. *If $x \equiv y$ and $z \in [x]$ then $z \in [y]$.*

Proof: (1) Follows from theorem (12). (2) Follows from **Q12** and **Q2**.▮

Now we can state the main result of this section:

Theorem 13 *With the usual restrictions, $\forall x \forall y (x \equiv y \Rightarrow (A(x, x) \Rightarrow A(x, y)))$.*

Proof: Suppose $m(x)$. From **Q9**, $m(y)$. In this case, $A(x, x)$ is only built from the following types of atomic formulas for some qset w: (1) $x \in w$; (2) $[x] \in w$; (3) $[x] \subseteq w$; (4) $qc([x]) =_E \lambda$; (5) $z \in [x]$. From lemmas (7), (8), and (9), we have: (1) $y \in w$; (2) $[y] \in w$; (3) $[y] \subseteq w$; (4) $qc([y]) =_E \lambda$; (5) $z \in [y]$, i.e., if $A(x, x)$ then $A(x, y)$. Suppose now that $\neg m(x)$. According to **Q9** $\neg m(y)$. By means of **Q4#**, if $A(x, x)$ then $A(x, y)$.▮

This last theorem says that indistinguishability $\equiv$ collapses into identity in $\mathcal{Q}\#$. So, $\mathcal{Q}\#$ is equivalent to standard ZFU.

This last theorem depends essentially on the Weak Axiom of Extensionality **Q26**. It is worth to remark that axiom **Q4#** was used by one of us in [11]. But in that paper the Axiom of Extensionality was a little bit different, so, some results presented here are not valid in the quasi-set theory introduced in [11].

In the next section, we shall present some ideas relating quasi-set theory and physics. The contents of this section are also discussed in [24].

4 Physics: The Maxwell-Boltzmann Statistics

According to usual textbooks on statistical mechanics, Maxwell-Boltzmann (MB) statistics gives us the most probable distribution of N *distinguishable* objects into, say, boxes with a specified number of objects each. In this section we show that the hypothesis concerning the objects being distinguishable is unnecessary. To do so, we need to recall some results from standard ZF.

4.1 Some Standard Results in ZF

In Zermelo-Fraenkel set theory the following results are proven without difficulty (in order to not confuse the considered 'sets' with those of quasi-set theory, we shall refer to them as 'ZF-sets'):

Lemma 10 *If x is a finite ZF-set, then*

$$card(\mathcal{P}(x)) = 2^{card(x)}.$$

Theorem 14 *Let x be a non-empty and finite ZF-set. If we define x_2 as a set of ordered pairs $\langle y_1, y_2 \rangle$ such that $y_1, y_2 \in \mathcal{P}(x)$, $y_1 \cup y_2 = x$, and $y_1 \cap y_2 \emptyset$ then $card(x_2) = 2^{card(x)}$.*

This theorem corresponds to say that the number of ways we can distribute N distinguishable particles ($N = card(x)$) between *two* boxes (represented by the ordered *pair* $\langle y_1, y_2 \rangle$) is 2^N.

Theorem 15 *Let x be a finite ZF-set such that $card(x) = N$. If we define x_n as a set of ordered n-tuples $\langle y_1, \cdots, y_n \rangle$ such that for all $i = 1, \cdots, n$ we have $y_i \in \mathcal{P}(x)$, $\bigcup_i y_i = x$, and $i \neq j \Rightarrow y_i \cap y_j = \emptyset$, then $card(x_n) = n^N$.*

We could rewrite theorem (15) as:

Theorem 16 *Let x be a finite ZF-set such that $card(x) = N$. If we define x_n as a set of ordered n-tuples $\langle y_1, \cdots, y_n \rangle$ such that for all $i = 1, \cdots, n$ we have $y_i \in \mathcal{P}(x)$, $\bigcup_i y_i = x$, and $\sum_i card(y_i) card(x)$, then $card(x_n) = n^N$.*

Proof: Analogous to the proof of theorem (15), since $\bigcup_i y_i = x$, and $ii \neq j \Rightarrow y_i \cap y_\emptyset$ iff $\bigcup_i y_i = x$, and $\sum_i card(y_i) card(x)$.▮

This theorem corresponds to say that the number of ways that we can distribute N distinguishable particles ($N = card(x)$) among n boxes (represented by the ordered n-tuple $\langle y_1, \cdots, y_n \rangle$) is n^N.

4.2 Quasi-Set-Theoretical Combinatorics

Given these results, we remark that we can obtain a perhaps more fruitful quasi-set theory, which indices a 'quasi-set-theoretical combinatorics' by exchanging the axiom **Q25** by the following postulate, which is a generalization of **Q25**), as well as a quasi-set-theoretical version of theorem (15):

Q25' Let x be a finite quasi-set such that $qc(x) =_E N$. If we define z_n as the quasi-set whose elements are ordered n-tuples $\langle y_1, \cdots, y_n \rangle$, where, for all $I =_E 1, \cdots, n$, we have $y_i \in \mathcal{P}(x)$, $\bigcup_i y_I x$, and $\sum_i qc(y_i) =_E qc(x)$, then we have the following:

$$qc(z_n) =_E n^N. \tag{4}$$

In the case where $n =_E 2$, we have a sentence which is equivalent to axiom **Q25**.

The aim of axiom **Q25'** is to allow us to define a quasi-set theoretical combinatorics which can be useful to cope with distribution functions. From the mathematical point of view, it is important to show that the exchange of axiom **Q25** by **Q25'** does not entail inconsistencies in quasi-set theory, supposing this theory is consistent. This shall be proven in the Section 5. The point, at this moment, is that **Q25** is very 'weak' if we are interested on a quasi-set-theoretical combinatorics with more than two physical states or 'boxes', as exemplified in the Introduction. Besides, axiom **Q25'** is our quasi-set theoretical version of theorem (16).

If we recall the polynomial of Leibniz, we can rewrite equation (4) as:

$$qc(z_n) =_E n^N =_E \sum \frac{N!}{\Pi_{i=1,\cdots n} n_i!}, \tag{5}$$

where the sum is over all possible combinations of nonnegative integers n_i such that $\sum_{i=1,\cdots,n} n_I =_E N$.

Interpreting n as the number of physical states, N as the total number of particles and n_i as the number of particles associated to each physical state i, then it is easy to see that each parcel of the summation in equation (5) is a possible MB (Maxwell-Boltzmann) distribution of N particles in n possible states. The most probable among all these parcels is precisely the MB distribution. So, we can add equation (5), with its respective interpretation, as another extra-assumption (an 'empirical axiom') in quasi-set theory. In other words, we are generalizing theory $\mathcal{Q}$, by replacing axiom **Q25** by axiom **Q25'**. We refer to this generalized quasi-set theory as $\mathcal{Q}'$. If we do not replace axiom **Q25** by **Q25'**, there is no way of saying anything about a distribution of N particles among an arbitrary number n of states or boxes. In this case, we would be confined to the very particular case of 2 states only.

It is easy to see that, for all i, we have $n_i =_E qc(y_i)$. Axiom **Q25'** is just another way of saying that the number of ways we can distribute N objects (either distinguishable or not) among n boxes is n^N. The condition that $\bigcup_i y_i =_E x$, and $\sum_i qc(y_i) =_E qc(x)$ is simply a way to guarantee that there will be no 'repeated occurrence' of the same object in two boxes. Nevertheless, it is obvious that the expression 'repeated occurrence', in this quasi-set-theoretical context, is just an intuitive approach for didactic purposes, since there is no sense in saying that the 'same' object cannot occupy two boxes.

The reader could ask: what are the so-called "boxes"? Each y_i corresponds to a given box or physical state. There can be, of course, two indistinguishable boxes y_i and y_j. In this case, the labels i and j cannot individualize each box. They are just different names, or labels, attributed to two indistinguishable objects (qsets, in this case).

4.3 One Simple Example

Now, let us exhibit an example in order to illustrate our ideas. Consider a collection of three indistinguishable particles to be distributed in two possible states or 'boxes'. According to standard textbooks on statistical mechanics, there are only four possibilities of distribution. On the other hand, according to our axiomatic framework – axiom **Q25'** – there are eight possibilities. If we impose that the occupation number of each box is constant, the number of possibilities corresponds to one parcel of the sum in equation (5).

The question now is: what about the extra four possibilities predicted by axiom **Q25'**? The eight possibilities predicted by **Q25'** and equation (5) come from

$$2^3 = \frac{3!}{3!0!} + \frac{3!}{2!1!} + \frac{3!}{1!2!} + \frac{3!}{0!3!}.$$

So, we have one possibility with 3 particles in the first state and no particle in the second state, plus three *indistinguishable* possibilities with 2 particles in the first state and 1 particle in the second state, plus three *indistinguishable* possibilities with 1 particle in the first state and 2 particles in the second state, plus one single possibility with no particle in the first state and 3 particles in the remaining one. The calculation of the most probable case is made for a large number of particles, following the standard calculations of statistical mechanics.

Following our example, axiom **Q25'** says that we can distribute 3 objects (either indistinguishable or not) among 2 boxes in 2^3 ways (either indistinguishable or not). But this axiom does not say *how* can we make this distribution. If we do not appeal to equation (5), we have the following: according to Fig. 1, there are, at least, by means of axiom **Q16**, *four* possible distributions. But axiom **Q25'** says that there are eight possible distributions. One possibility is something like Fig. 2, that is, the four distributions in Fig. 1 *plus* four distributions which are indistinguishable from the third distribution of Fig. 1. The reader can easily imagine other possibilities. So, axiom **Q25'** by itself does not allow us to derive MB statistics. It simply says that MB statistics is a possibility even in a collection of indiscernible objects. Axiom **Q25'** *and* equation (5), with its respective interpretation in the context of **Q25'**, is a way to say that the only possibility is that one illustrated at the Fig. 3.

5 Quantum Statistics

Since we may have MB distribution among non-individuals, what is the difference between quantum statistics and MB, after all? In Bose-Einstein statistics, we take into account *only distinguishable possibilities*, among all possibilities predicted by axiom **Q25'**. Fermi-Dirac statistics is derived in the same vein, but with the additional assumption of the quasi-set theoretical version of Pauli's Exclusion Principle: $qc(y_i) \leq 1$ for each i in **Q25'**. Putting it another way, quantum statistics may be seen as special cases of MB statistics of a collection of indistinguishable particles.

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Figure 1: The ‘first’ four possible distributions of 3 objects (indistinguishable or not) among 2 boxes. Each line represents one possible distribution and each bullet represents an object.

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Figure 2: One possible sequence of the eight possible distributions of 3 objects among 2 boxes according to axiom **Q25’**.

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Figure 3: The only possible distribution of 3 objects among 2 boxes, if we conjugate axiom **Q25’** and equation (2).