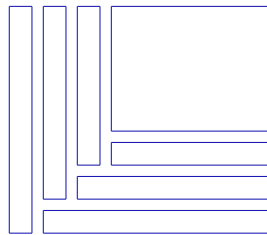

Basic Ideas of Quasi-Set Theory

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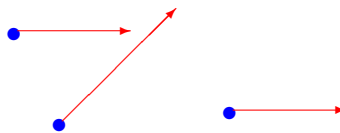
DEDICATED TO NEWTON DA COSTA, TEACHER AND FRIEND, FOR HIS
80TH BIRTHDAY.

—August 2009—

CLE-AIPS Event —“Science, Truth, and Consistency”

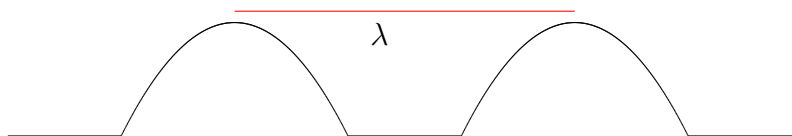
1 Motivations

- In the beginnings of quantum physics, Heisenberg, Born, Schrödinger, Bohr, later Weyl, Hesse, etc. spoke of the *lost of individuality* of quantum entities.
- They have also referred to these entities as *non-individuals*.
- In fact, there seem to be no differences among atoms of the same substance, so as among other sub-atomic ‘particles’ of the same kind. In a certain sense, all protons (electrons, neutrons, quarks, ...) are *exactly* alike.
- The main motivation is of course a possible way to interpret quantum objects.
- ‘Quantum objects’ may be particles (as in orthodox QM), fields, whatever you mean.
- Think for instance in a BEC (**Bose-Einstein Condensate**).
 - **Classical Physics**: particles (like small balls) are characterized by their *position* and *velocity*.



- **Quantum Mechanics**: the particle-wave duality. They propagate as waves, but we detect particles (they make ‘clicks’ in the detectors). The de Broglie wavelength is inversely proportional to the velocity:

$$\lambda \approx \frac{h}{m.v}$$



- Mater waves:

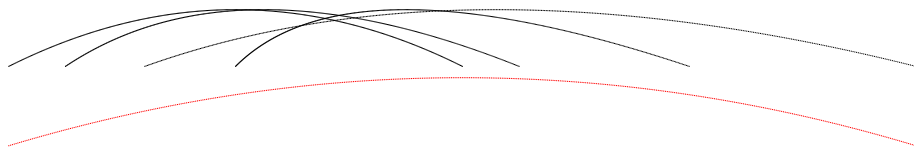
$$\lambda \approx \frac{h}{mv} \quad \lambda \approx (\dots) \frac{1}{\sqrt{T}}$$

$$\underbrace{m \cdot v^2}_{\text{kinetic energy}} = \underbrace{k \cdot T}_{\text{thermal energy}}$$

- Temperature vs. wave lenght

T	λ	
here	$10^{-10}m = 1\text{\AA}$	$T \downarrow, \lambda \uparrow$
$1\mu K$	$10^{-6}m = 1\mu m$	
$1nK$	$30\mu m$	

- BEC As the temperature becomes ‘critic’ (*circa* few pico Kelvins), the wavelenghts become longer, so that the ‘particles’ cannot be followed as individuals any more: they became a *soup* of matter waves.



- But we may associate an interpretation to the phenomena; thinking of the waves representing quantum entities, they are of course not *one* ‘object’. How can we speak of *individuals* here?
- Can they be seen as distinct *solo numero*? (Permutations of objects of the same kind lead to the same physical significative values).

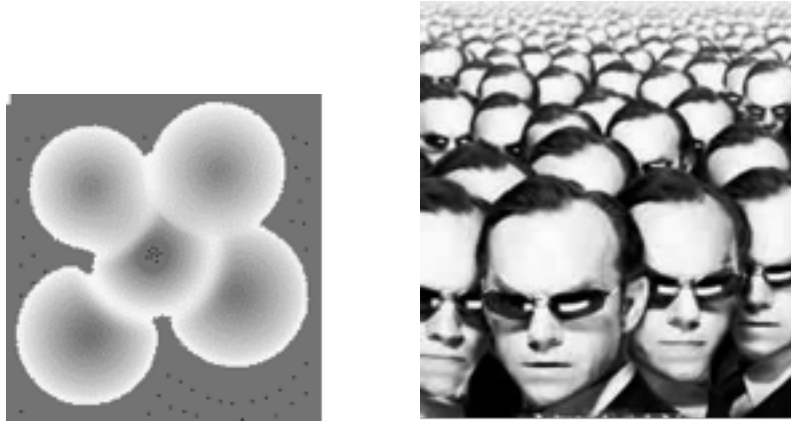


Figure 1: Are there differences between cold atoms and the 100 Smiths?

- What's the difference between some cold atoms and the 100 Smiths? Are they *different*? In what sense?
- We can pursue (formally) a *metaphysics of non-individuals*, grounded on a possible interpretation of quantum 'objects'. Informally speaking, these are objects to which the standard notion of identity does not apply.
- *Why?* If the standard theory of identity holds for some objects, they can always (in principle) be discerned from any other object. They become *individuals*.
- Classical theory of identity says that indiscernible things are the very same thing.
 - There are not indistinguishable but not identical objects.
- *Indiscernibility* may be assumed to be a fundamental concept (in QM applications, I shall avoid discussing interpretations such as Bohm-Hiley's –but see French & Krause [3]).
- Heinz Post (1963): the indiscernibility (non-individuality) of quantum objects should be considered *right at the start* (as a primitive notion).

- Usually, the formalism of orthodox QM uses **symmetrization postulates**: symmetric and anti-symmetric vectors/functions express indiscernibility.
- For two systems 1 and 2 (**we need to label them**—our languages are *objectual* languages) and two possible states a and b , we have

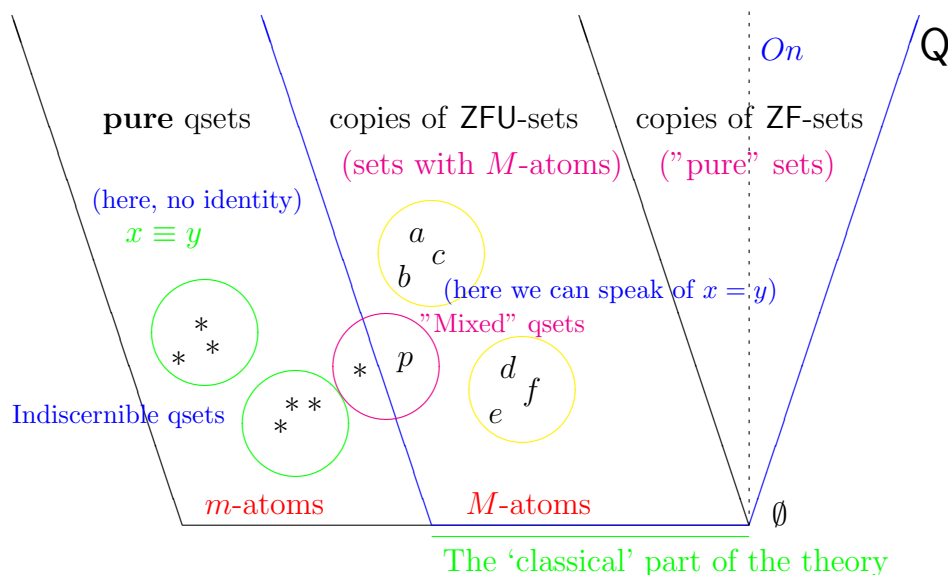
$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|\psi_1^a\rangle|\psi_2^b\rangle \pm |\psi_2^a\rangle|\psi_1^b\rangle)$$

$$\langle\psi_{12}|\hat{O}|\psi_{12}\rangle = \langle\psi_{21}|\hat{O}|\psi_{21}\rangle$$

The expectation value of the measure of any observable does not change after a permutation of the particles.

- We should try to find a formalism (a logic) for QM without appealing to artificial labeling of quanta. And to find a mathematical framework to found a semantics for such a formalism, grounded on the metaphysics of non-individuals.
- da Costa (1980)—von Weizsäcker (1971)'s *principle of semantic consistency*: “the rules by which we describe and guide our measurement, defining the semantics of the formalism of a theory, must be in accordance with the laws of the theory.” (von Weizsäcker)

2 The quasi-set universe



On is the class of ordinals, defined in the ‘classical’ part of the theory. This ‘classical part’ contains a copy of ZFU—and of ZF. Those qsets whose transitive closure don’t contain m -atoms are the “sets” of \mathfrak{Q} ; they obey the primitive predicate Z ($Z(x)$ says that x is a *set* in \mathfrak{Q}).

3 The quasi-set theory \mathfrak{Q}

3.1 The language of the formal theory

- (i) propositional connectives,
- (ii) quantifiers
- (iii) individual variables (a denumerable set)
- (iv) two binary predicates \equiv and \in ,
- (v) three unary predicates m , M and Z , and

(vi) an unary functional symbol qc .

Terms and formulas are defined as usual.

(i) $x \equiv y$ (x is indiscernible from y)

(ii) $m(x)$ (x is a "micro-object")

(iii) $M(x)$ (x is a "macro-object")

(iv) $Z(x)$ (x is a "set – a copy of a ZFU set)

(v) $qc(x)$ (the quasi-cardinal of x)

Definition 1

(i) $Q(x) =_{\text{def}} \neg(m(x) \vee M(x))$ (x is a qset)

(ii) $P(x) =_{\text{def}} Q(x) \wedge \forall y(y \in x \rightarrow m(y)) \wedge \forall y \forall z(y \in x \wedge z \in x \rightarrow y \equiv z)$

x is a pure qset, having m -atoms as elements, not necessarily indiscernible from one each other.

(iii) $D(x) =_{\text{def}} M(x) \vee Z(x)$

x is a *Ding*, a "classical object" in the sense of Zermelo's set theory, namely, either a set or a macro *Urelemente*.

(iv) $E(x) =_{\text{def}} Q(x) \wedge \forall y(y \in x \rightarrow Q(y))$

x is a qset whose elements are qsets.

(v) $x =_E y =_{\text{def}} (Q(x) \wedge Q(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge \forall_Q z(x \in z \leftrightarrow y \in z))$

Extensional identity)—we shall write simply $x = y$ instead of $x =_E y$ from now on.

(vi) $x \subseteq y =_{\text{def}} \forall z(z \in x \rightarrow z \in y)$ (subqset)

3.2 The postulates of \mathfrak{Q}

Postulates for first-order elementary logic without identity.

$$(\equiv_1) \quad \forall x(x \equiv x)$$

$$(\equiv_2) \quad \forall x \forall y(x \equiv y \rightarrow y \equiv x)$$

$$(\equiv_3) \quad \forall x \forall y \forall z(x \equiv y \wedge y \equiv z \rightarrow x \equiv z)$$

$$(\equiv_4) \quad \forall x \forall y(x = y \rightarrow (\alpha(x) \rightarrow \alpha(y)))$$

$$(\in_1) \quad \forall x \forall y(x \in y \rightarrow Q(y))$$

If something has an element, then it is a qset; in other words, the atoms have no elements (in terms of the membership relation).

$$(\in_2) \quad \forall_D x \forall_D y(x \equiv y \rightarrow x = y)$$

Indistinguishable *Dinge* are extensionally identical. This makes = and \equiv coincide for this kind of entities.

$$(\in_3) \quad \forall x \forall y[(m(x) \wedge x \equiv y \rightarrow m(y)) \wedge (M(x) \wedge x = y \rightarrow M(y)) \wedge (Z(x) \wedge x = y \rightarrow Z(y))]$$

$$(\in_4) \quad \exists x \forall y(\neg x \in y)$$

This qset will be proved to be a set (in the sense of obeying the predicate Z), and it is unique, as it follows from extensionality. Thus, from now on we shall denote it, as usual, by ' \emptyset '.

$$(\in_5) \quad \forall_Q x(\forall y(y \in x \rightarrow D(y)) \leftrightarrow Z(x))$$

This postulate grants that something is a set (obeys Z) iff its transitive closure does not contain m -atoms. That is, *sets* in \mathfrak{Q} are those entities obtained in the 'classical' part of the theory.

$$(\in_6) \quad \forall x \forall y \exists_Q z(x \in z \wedge y \in z)$$

(\in_7) If $\alpha(x)$ is a formula in which x appears free, then

$$\forall_Q z \exists_Q y \forall x (x \in y \leftrightarrow x \in z \wedge \alpha(x)).$$

This is the Separation Schema; notation

$$[x \in z : \alpha(x)].$$

When this qset is a set, we write, as usual, $\{x \in z : \alpha(x)\}$.

(\in_8) $\forall_Q x (E(x) \rightarrow \exists_Q y (\forall z (z \in y \leftrightarrow \exists w (z \in w \wedge w \in x))))$.

The union of x , written $\cup x$. Usual notation is used in particular cases.

3.3 Some basic concepts

From (\in_6): $\forall x \forall y \exists_Q z (x \in z \wedge y \in z)$

Using $\alpha(w) \leftrightarrow w \equiv x \vee w \equiv y$, we get a subqset of z which we denote

$$[x, y]_z$$

which is the qset of the indiscernibles of either x or y that belong to z .

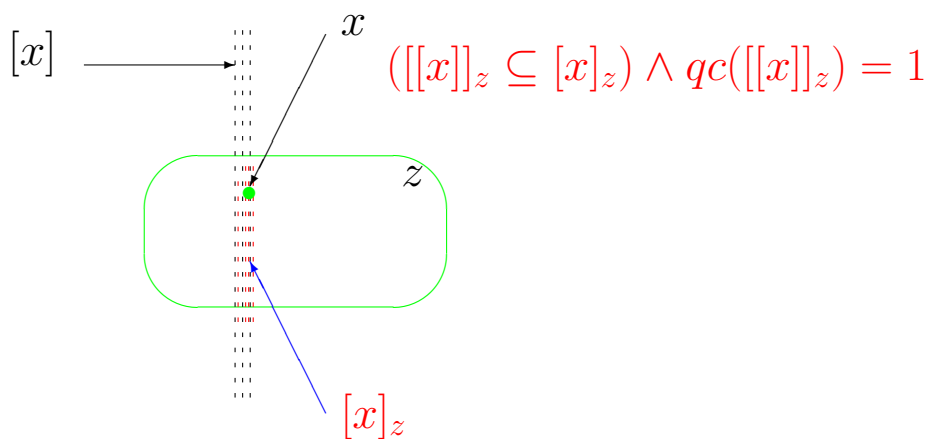
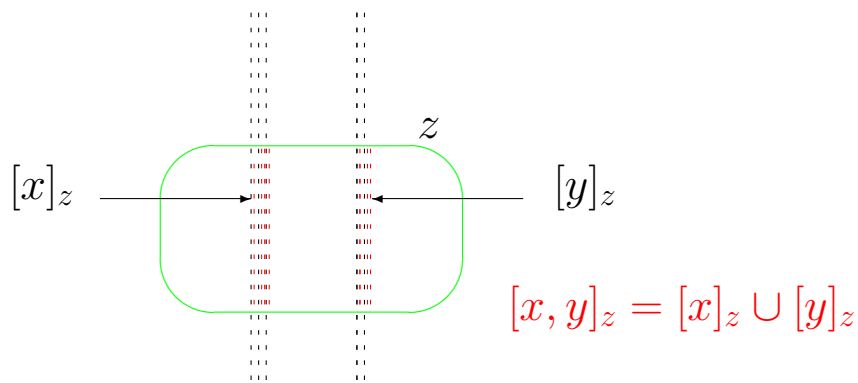
When $x \equiv y$, this qset reduces to

$$[x]_z$$

called the qset of the indiscernible from x that belong to z .

The qset $[x, y]_z$ does not have necessarily *two* elements (that is, we may have $qc([x, y]_z) > 2$).

The qset z and one of its elements, x . The collections $[x]$ and $[x]_z$ stand for *all* indiscernible from x and the qset of the



indiscernible from x that belong to z respectively. (Usually, $[x]$ is too big to be a qset.)

Later, with the postulates of quasi-cardinal, we will be able to prove $[x]_z$ has a subqset with quasi-cardinal equals to 1:

$$[[x]]_z$$

We call it the **strong singleton** of x (really, *a* strong singleton of x , for we cannot grant that it is unique). It has just one element, and we can think that this element *as if* it is x .

Really, $[[x]]_z$ contains *one object of the 'species' x* . That is, $qc([[x]]_z) = 1$.

3.4 Other postulates and definitions

$$(\in_9) \forall_Q x \exists_Q y \forall z (z \in y \leftrightarrow w \subseteq x),$$

The power qset of x , denoted $\mathcal{P}(x)$.

$$(\in_{10}) \forall_Q x (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup [y]_x \in x)),$$

The infinity axiom.

$$(\in_{11}) \forall_Q x (E(x) \wedge x \neq \emptyset \rightarrow \exists_Q y (y \in x \wedge y \cap x = \emptyset)),$$

The axiom of foundation, where $x \cap y$ is defined as usual.

Definition 2 (Weak ordered pair)

$$\langle x, y \rangle_z =_{\text{def}} [[x]_z, [x, y]_z]_z \quad (1)$$

Then, $\langle x, y \rangle_z$ takes all indiscernible from either x or y that belong to z , and it is called the “weak” ordered pair, for it may have more than two elements. Sometimes the sub-indices z will be left implicit.

Definition 3 (Cartesian Product) *Let z and w be two qsets. We define the cartesian product $z \times w$ as follows:*

$$z \times w =_{\text{def}} [\langle x, y \rangle_{z \cup w} : x \in z \wedge y \in w] \quad (2)$$

Functions and relations cannot also be defined as usual, for when there are m -atoms involved, a mapping may not distinguish between arguments and values. Thus we provide a wider definition for both concepts, which reduce to the standard ones when restricted to classical entities. Thus,

Definition 4 (Quasi-relation) *A qset R is a binary quasi-relation between to qsets z and w if its elements are weak ordered pairs of the form $\langle x, y \rangle_{z \cup w}$, with $x \in z$ and $y \in w$.*

Definition 5 (Quasi-functions) *Quasi-functions: injective, so-brejective, bijective.*

3.5 Postulates for quasi-cardinals

Here α, β, \dots stand for cardinals (defined in the classical part of the theory):

$$(qc_1) \quad \forall_Q x (\exists_Z y (y = qc(x)) \rightarrow \exists! y (Cd(y) \wedge y = qc(x) \wedge (Z(x) \rightarrow y = card(x))))$$

If the qset x has a quasi-cardinal, then its (unique) quasi-cardinal is a cardinal (defined in the ‘classical’ part of the theory) and coincides with the cardinal of x stricto sensu if x is a set.

$$(qc_2) \quad \forall_Q x (x \neq \emptyset \rightarrow qc(x) \neq 0).$$

Every non-empty qset has a non-null quasi-cardinal.

$$(qc_3) \quad \forall_Q x (\exists_Z \alpha (\alpha = qc(x)) \rightarrow \forall \beta (\beta \leq \alpha \rightarrow \exists_Q z (z \subseteq x \wedge qc(z) = \beta)))$$

If x has quasi-cardinal α , then for any cardinal $\beta \leq \alpha$, there is a sub-qset of x with that quasi-cardinal.

In the remaining axioms, for simplicity, we shall write $\forall_{Q_{qc}} x$ (or $\exists_{Q_{qc}} x$) for quantifications over qsets x having a quasi-cardinal.

$$(qc_4) \quad \forall_{Q_{qc}} x \forall_{Q_{qc}} y (y \subseteq x \rightarrow qc(y) \leq qc(x))$$

$$(qc_5) \quad \forall_{Q_{qc}} x \forall_{Q_{qc}} y (Fin(x) \wedge x \subset y \rightarrow qc(x) < qc(y))$$

It can be proven that if both x and y have a quasi-cardinal, then $x \cup y$ has a quasi-cardinal. Then,

$$(qc_5) \quad \forall_{Q_{qc}} x \forall_{Q_{qc}} y (\forall w (w \notin x \vee w \notin y) \rightarrow qc(x \cup y) = qc(x) + qc(y))$$

In the next axiom, $2^{qc(x)}$ denotes (intuitively) the quantity of subquasi-sets of x . Then,

$$(qc_6) \quad \forall_{Q_{qc}} x (qc(\mathcal{P}(x)) = 2^{qc(x)})$$

3.6 The Weak Extensionality Axiom

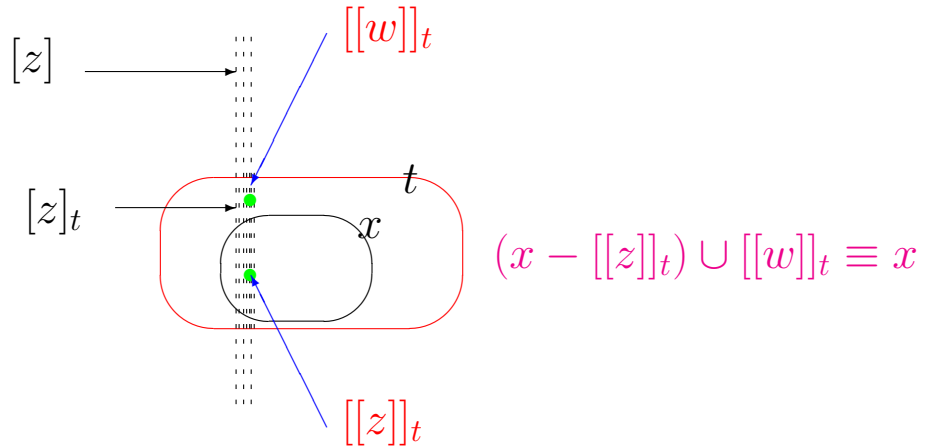
$$(\equiv_1 2) \forall_Q x \forall_Q y ((\forall z (z \in x / \equiv \rightarrow \exists t (t \in y / \equiv \wedge \wedge_Q Sim(z, t)))) \wedge \forall t (t \in y / \equiv \rightarrow \exists z (z \in x / \equiv \wedge \wedge_Q Sim(t, z))) \rightarrow x \equiv y)$$

The following theorem express the invariance by permutations in \mathfrak{Q} , and with this result we finish our revision:

Theorem 1 (*Invariance by Permutations*) *Let x be a finite qset such that $\neg(x = [z]_t)$ for some t and let z be an m -atom such that $z \in x$. If $w \in t$, $w \equiv z$ and $w \notin x$, then there exists $[[w]]_t$ such that*

$$(x - [[z]]_t) \cup [[w]]_t \equiv x$$

This above theorem is illustrated by the figure ?? below, where $[z]$ is the collection—or “quasi-class”—of all indiscernibles of z , while $[z]_t$ is given by the pair axiom and the separation schema:



The invariance by permutations in \mathfrak{Q} . Two indiscernible elements from $z \in x$ and $w \notin x$, expressed by their quasi-singletons

$[[z]]_t$ and $[[w]]_t$, are “interchanged” and the resulting qset x remains indiscernible from the original one. The hypothesis that $\neg(x = [z]_t)$ grants that there are indiscernible from z in t which do not belong to x .

4 Relative Consistency

It is easy to see (as put by French & Krause [3]) that there is a translation from the language of ZFU (\mathcal{L}_{ZFU}) to the language of \mathfrak{Q} , so that if \mathfrak{Q} is consistent, so is ZFU (and hence so is ZF). The translation can be described as follows, if we suppose that \mathcal{L}_{ZFU} has a primitive unary symbol S so that $S(x)$ says intuitively that x is a set. Then, being A any formula of the language of ZFU, let A^q be its translation in the language of \mathfrak{Q} , defined as follows:

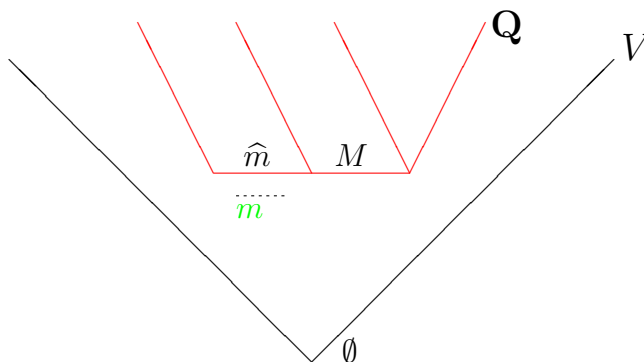
Definition 6

- (a) If A is $S(x)$, then A^q is $Z(x)$
- (b) If A is $x = y$, then A^q is $((M(x) \wedge M(y)) \vee (Z(y) \wedge Z(y))) \wedge x =_E y$
- (c) If A is $x \in y$, then A^q is $((M(x) \vee Z(x)) \wedge Z(y)) \wedge x \in y$
- (d) If A is $\neg B$, then A^q is $\neg B^q$
- (e) If A is $B \vee C$, then A^q is $B^q \vee C^q$
- (f) If A is $\forall x B$, then A^q is $\forall x (M(x) \vee Z(x) \rightarrow B)$
- (g) Finally, the term $qc(x)$ is interpreted in $card(x)$.

It is immediate that $\text{Cons}(\mathfrak{Q}) \mapsto \text{Cons}(\text{ZFU})$.

Now let us consider in brief the converse of this result.

We shall define a superstructure \mathbf{Q} over a certain set X . It results that \mathbf{Q} is a 'model' for \mathfrak{Q} .



The elements of m —dashed lines—are outside \mathbf{Q} . Thus, *within* the structure \mathbf{Q} , we cannot “see” that the elements of m are individuals.

In ‘pure’ ZF (without *Urelemente*):

(1) $m \neq \emptyset$ a non empty set

(2) R is an equivalence relation on m .

The equivalence classes of the quotient set m/R are denoted by C_1, C_2, \dots

(3) If $x \in m$, define

$$\hat{x} = \langle x, C_x \rangle,$$

where C_x is the equivalence class to which x belongs and call \hat{m} the set of all \hat{x} with $x \in m$.

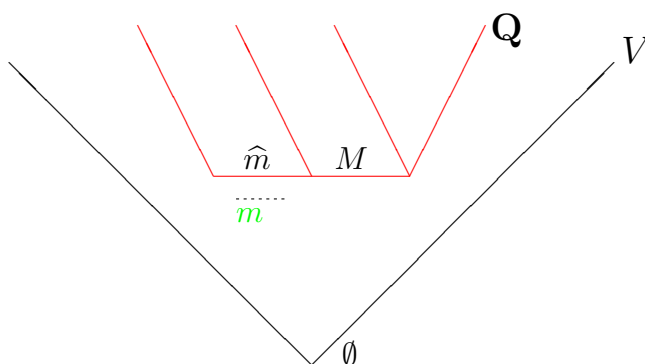
The basis:

$$X = \hat{m} \cup M,$$

where \hat{m} is as above and M is a set such that $\hat{m} \cap M = \emptyset$ and $rank(\hat{m}) = rank(M)$.

The definition is as follows:

$$\begin{aligned}
 Q_0 &=_{\text{def}} X \\
 Q_1 &=_{\text{def}} X \cup \mathcal{P}(X) \\
 &\vdots \\
 Q_\lambda &=_{\text{def}} \bigcup_{\beta < \lambda} Q_\beta \quad \text{if } \lambda \text{ is a limit ordinal} \\
 \mathbf{Q} &=_{\text{def}} \bigcup_{\alpha \in O_n} Q_\alpha.
 \end{aligned}$$



4.1 Some further questions

- Apparently, \mathbf{Q} is not rigid.
- Apparently, if we postulate the existence of m -objects, \mathbf{Q} cannot be embedded within ZF.

5 Applications to the foundations of QM

- **Aim:** To find a ‘new interpretation’ of the standard formalism which may involve non-individuality ‘right from the start’.

— In Domenech et al. [1] and [2] we have used the non-classical part of \mathbf{Q} to define a Hilbert space (called \mathbf{Q} -space) whose vectors refer only to occupation numbers, while permutation operators act as the identity operator on them, reflecting in the formalism the unobservability of permutations.

By maintaining both quantum indistinguishability and antisymmetry without resort to a symmetrization postulate, we derived spin values in a two-value fermionic system, obtaining identical results to the Fock space formalism **without appeal to artificial labeling of quanta**.

Perhaps these are the first steps in founding a QM with truly indiscernible objects.

- To develop a mathematics having motivations in these ideas, including the correspondent mathematics used in quantum physics.

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