Logical Aspects of Quantum (Non-)Individuality

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1 Indiscernibility in QM

In the standard formalism, indiscernibility is *made by hand*, by means of symmetric and anti-symmetric functions/vectors.

We start with labeling quanta, say by $x_1, \ldots, x_n$, and then we use permutational symmetry to express the indiscernibility:

If F is an n-ary predicate, then $F(x_1, \ldots, x_n)$ is equivalent to $F(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for every permutation π defined on $\{1, \ldots, n\}$.

Well known examples: the symmetric and anti-symmetric vector for a system with two quanta:

$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|\psi_1{}^1\rangle|\psi_2{}^2\rangle + |\psi_2{}^1\rangle|\psi_1{}^2\rangle)$$

$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|\psi_1{}^1\rangle|\psi_2{}^2\rangle - |\psi_2{}^1\rangle|\psi_1{}^2\rangle)$$

Invariance by permutations:

$$||\psi_{12}\rangle|^2 = ||\psi_{21}\rangle|^2$$

What does it mean?

2 More precisely: indiscernibility in a structure

The additive group of the integers, $\mathcal{Z} = \langle \mathbb{Z}, + \rangle$.

This structure has two automorphisms

(i) The identity function $i(x) = x$ and

(ii) The "opposite map" $h(x) = -x$.

Two objects a and b in the domain of a certain structure $\mathcal{A}$ are indiscernible in $\mathcal{A}$ (or are $\mathcal{A}$-indiscernible) if there exists an automorphism f of $\mathcal{A}$ such that $f(a) = b$.

Thus, 2 nd -2 are $\mathcal{Z}$-indiscernible, although they are of course not identical.

The distinction between 2 and -2 cannot be seen *from the inside* of the structure $\mathcal{Z}$.

Take a larger structure, say $\mathcal{Z}' = \langle \mathbb{Z}, +, \{n\}_{n \in \mathbb{Z}} \rangle$. In this extended structure, 2 and -2 can be distinguished for $2 \in \{2\}$, but $-2 \notin \{2\}$.

When the only automorphism of a structure is the identity function, as in $\mathcal{Z}' = \langle \mathbb{Z}, +, \{n\}_{n \in \mathbb{Z}} \rangle$, we say that the structure is *rigid*, that is, its Galois group is the trivial group.

∗ Theorem: In ZF (Zermelo-Fraenkel set theory) *any structure can be extended to a rigid structure.*

ZF, as a whole, is rigid [the well-founded universe $\mathcal{V} = \langle V, \in \rangle$] is rigid.

What about QM? (Non-relativistic quantum theory only, but in principle the ideas hold in general), we can consider it as characterized by structures like

$$\mathcal{Q} = \langle F, S, Q_0, \ldots, Q_n, \rho \rangle \quad (1)$$

where F is a mathematical model of standard functional analysis, S is a set of "physical situations", $Q_0, \ldots, Q_n$ represent the observational part of the theory (physical observables) and ρ is a mapping which ascribes an element of F to each $s \in S$ (an adequate Hilbert space), and an Hermitian operator on a suitable Hilbert space to each Q_i $(i = 1, \ldots, n)$.

Thus the postulates of QM are:

- (1) the logical postulates (say, first-order logic with identity)
- (2) the "mathematical" postulates (say, those of ZF)
- (3) the specific postulates (those of QM proper)

We would not forget that the postulates of QM encompass also (1) and (2).

Hence the above $*$Theorem is a theorem of QM!! ($*$Theorem: In ZF (Zermelo-Fraenkel set theory) *any structure—including* $\mathcal{Q}$*—can be extended to a rigid structure.*)

Thus, within the scope of standard mathematics (read: that one that can be "constructed" within ZF) we can always distinguish between two distinct objects whatever, if not in the considered structure, then in some of its rigid extensions.

In ZF (and within the mathematics built in ZF), there are not *genuine* indiscernible objects (objects which differ *solo numero*).

But $\mathcal{Q}$ is an structure in ZF, so, it can be extended to a rigid structure where all (representable) objects are individuals.

The most we can do is to talk of objects indiscernible with respect to a structure.

So, we could enlarge the usual proposes for a principle of individuation: standardly, we have:

- some kind of substratum (haecceity, thisness, etc.)
- some property/relation (or a set of them) –Bundle Theories
- the underlying logic. For instance, classical logic entails individuality.

Why?

3 The classical theory of identity

Intuitively speaking, $a = b$ means that the objects denoted by a and by b are *the very same* object.

Set-theoretically (in extensional set theories), the identity of (the set) A is the binary relation (called the *diagonal* of A):

$$\Delta_A =_{\mathsf{def}} \{\langle x, x\rangle : x \in A\}, \tag{2}$$

Can we axiomatize Δ_A?

3.1 First-order languages

Being $=$ a binary primitive predicate symbol,

$(=_1)$ $\forall x(x = x)$

$(=_2)$ $\forall x \forall y(x = y \to (\alpha(x, x) \to \alpha(x, y))$, where $\alpha(x, x)$ is any formula where x appears free and $\alpha(x, y)$ is obtained from $\alpha(x, x)$ by the substitution of some (not necessarily all) free occurrences of x by y, and y is free for x in $\alpha(x, x)$.

These axioms do not axiomatize identity, in the sense of not fixing the diagonal of the domain. They axiomatize a congruence relation, but not necessarily identity. Why?

Suppose that $\mathfrak{A} = \langle D, \rho \rangle$ is, as above, an interpretation for our first-order language, where $D \neq 0$ and ρ is the denotation function.

Let us call Δ_D the diagonal of D. Now let $\sim$ be any equivalence relation defined on D, and let us take the set $D^* =_{\mathsf{def}} D/\sim$ as the domain of a new interpretation $\mathfrak{A}^* = \langle D^*, \rho^* \rangle$ for $\mathcal{L}$.

We can prove that there exists a mapping $f : D \mapsto D^*$ defined as follows (I shall assume that $\mathcal{L}$ has no functional symbols for simplicity):

(i) For every $x \in D$, $f(x)$ is the equivalence class in D^* to which x belongs.

(ii) For every x and y in D, $\langle f(x), f(y) \rangle \in \Delta_{D^*}$ iff $\langle x, y \rangle \in \Delta_D$.

(iii) For every n-ary predicate symbol P of $\mathcal{L}$, $\langle f(x_1), \ldots, f(x_n) \rangle \in \rho(P)$ iff $\langle f(x_1), \ldots, f(x_n) \rangle \in \rho^*(P)$

(iv) For every individual constant a, $\rho^*(a) = f(\rho(a))$.

The structures $\mathfrak{A}$ and $\mathfrak{A}^*$ are *elementary equivalent*: the sentences of $\mathcal{L}$ which are true in $\mathfrak{A}$ are the same as those which are true in $\mathfrak{A}^*$.

That is, the two structures cannot be distinguished by means of the resources in $\mathcal{L}$.

Hence we cannot know if we are speaking of individuals of the domain or of equivalence classes of these individuals.

4 Defining identity

In higher-order languages (say, second order, which suffices for the argumentation) we can define identity by means of Leibniz Law:

$$x = y =_{\mathsf{def}} \forall F(F(x) \leftrightarrow F(y)), \tag{3}$$

where x and y are individual variables and F is a variable ranging the set of the properties of these individuals.

Higher order languages are of course stronger than first order languages. In particular, we can define identity as (3) and not merely a congruence relation, as first-order axioms do.

But in order to ensure that Leibniz Law grants that two individuals a and b are *really* identical, we need to work with full (standard) models, that is, structures that encompass *all* the sub-sets of the domain of the interpretation.

If not, that is, if we consider Henkin-style models, it is easy to show that we can present situations where a and b satisfy (3) and even so are distinct:

For instance, let us take a domain $D = \{1, 2, 3, 4\}$ and suppose that the language has three unary predicates which are interpreted in the sub-sets $\{1, 2\}$, $\{1, 2, 3\}$ and $\{1, 2, 4\}$.

Furthermore, interpret a as 1 and b as 2. Then, of course a and b belong to the same sets (of the structure), that is, obey the same predicates, thus obeying (3), but 1 is not *identical* with 2.

In the standard extensional set theories, like first-order ZF (without *Urelemente*), in addition to the postulates ($=_1$) and ($=_2$), we add the Axiom of Extensionality:

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y). \tag{4}$$

But first order set theory (ZF is our paradigm), if consistent, raise the following questions:

- (1) Is there a model of (first-order) ZF that copes with reality? In this sense, there is a countable model too. So as models of every infinite cardinality.
- (2) Theories like ZF don't have intuitive models, except if we are strongly platonists like Gödel.
- (3) If even ZF has not intuitive model, how can such a theory provide us an intuitive model of reality?
- (4) Since we cannot characterize a standard model of (first order) ZF (that one where identity is truly identity), how can we be aware that finite numbers, which we use, for instance, in the Fock space formalism, are really finite?
- (5) In the standard Hilbert space formalism, we deal with bases for the relevant Hilbert spaces.
- (6) But in certain set theories in which the axiom of choice does not hold in full generality, we obtain:
  - (a) vector spaces with no basis, and
  - (b) vector spaces that have two bases of different cardinalities

- (7) In Solovay's models, every limited operator is bounded. But in QM we need unbounded operators (position, momentum).

Can we use such a mathematics to found QM?

5 First-order PII?

Consider the converse of $(=_2)$. The most we can do is to write it as an schema (in a denumerable language):

$$\forall x \forall x((\alpha(x, x) \to \alpha(x, y) \to x = y) \tag{5}$$

A particular case, where F is an unary predicate symbol.

$$\forall x \forall y(F(x) \to F(y)) \to x = y \tag{6}$$

Semantically, suppose that our domain is the set of natural numbers ω.

Then, according to standard semantics, to each F we associate a sub-set of ω.

But the number of sub-sets covered by (5) is at most $\aleph_0$, while ω has $2^{\aleph_0}$ sub-sets!

x and y can satisfy all those Fs in (5) and even so being not the same natural numbers.

Something seems to be missed here; if we restrict the predicates to a denumerable number only, are we admitting some kind of hidden predicates? —since the objects may agree with respect to all of them and even so being discernible...

This shows that with first-order PII, we cannot ensure that the objects that obey all the (at most) denumerable number of predicates or formulas are really identical.

In order to do so, we need to take *all* the predicates in order to be able to take *all* subsets of the domain and then to apply the set-theoretical postulates over the domain to see whether the objects are really identical.

Thus, to say that a and b are indiscernible by the predicate $=$ axiomatized by Frege's axioms is equivalent to say that a is identical to b ($a = b$), needs care, for it depends on what we understand by $a = b$.

If $a = b$ is taken as meaning that a and b are the very same object, then the affirmative may be false, as we have saw, since Frege's axioms do not characterize the diagonal of the domain.

The same happens with respect to their concept of physically indiscernible objects, which some people use as "PII in QM": if two physical systems are indiscernible with respect to a subset of predicates of the language of QM, then they are identical ($a = b$).

This can make sense from the physical point of view, but as far as the entities are described within ZF, physical indistinguishability does not entail identity.

Further, even when we say that a permutation of indiscernible quantum objects does not conduce to a different situation, we should add "physical situation", for from the perspective of classical logic of course changes were made by the simple fact that we may be exchanging two distinct objects, which violates the axiom of extensionality.

But even in first order languages, we can try to define identity if we can find a formula $\alpha(x, y)$ on two free variables x and y so that we can state that

$$x = y =_{\mathsf{def}} \alpha(x, y). \tag{7}$$

This is what happens, some people sustain, when we have a finite number of predicates, but does not hold in general if this is not the case. For instance, if our language has only the binary predicate P and the unary predicate Q, then we can take $\alpha(x, y)$ to be

$$\forall z((P(x,z) \leftrightarrow P(y,z)) \wedge (P(z,x) \leftrightarrow P(z,y))) \wedge (Q(x) \leftrightarrow Q(y)). \tag{8}$$

This definition is attributed to Hilbert and Bernays. Does it define identity? (in the sense of fixing the diagonal of the domain of an interpretation).

Not really.

It defines indiscernibility with respect to the chosen (considered) predicates of the language.

Quine acknowledges this: "It may happens that the objects intended as values of variables of quantification are not completely distinguished from one another by the (...) predicates. When this happens, (3) fails [his equation for (8) above] to define genuine identity. Still, such failure remains unobservable from within the language." (Quine [1986]–*Philosophy of Logic*, p.63)

6 Relational objects

a and *b* are *relational* if they satisfy an irreflexive (binary for simplicity) and symmetric relation R.

Ex.: two electrons in a singlet state obeying "... to have spin opposite from ...". (Muller & Saunders [2008])

[The Principle of the Absolute Indiscernibles, PII-A] No two objects can be *absolutely* indiscernible, that is, there is always some property (monadic predicate) that one of them has but the another does not.

[The Principle of the Relational Indiscernibles, PII-R] No two objects are relational indiscernible, that is, there is always a relational property that distinguishes them.

[The Principle of the Identity of Indiscernibles, PII] No two objects are absolutely and relationally indiscernible

(i) PII-A $\rightarrow$ PII (from the definition)

(ii) PII-R $\rightarrow$ PII (idem)

(iii) PII-A $\rightarrow$ PII-R (for absolute discernible objects are always relational discernible—Muller & Saunders [2008], pp.528-9)

(iv) Hence PII $\leftrightarrow$ PII-R

(v) Muller and Saunders claim that $\neg$(PII $\rightarrow$ PII-A), and then

(vi) PII $\wedge$ $\neg$PII-A

(vii) So, PII-R $\wedge$ $\neg$PII-A (by iv and vi)

But it is easy to see that PII-R $\rightarrow$ PII-A.

The first argument comes from the above discussion about the extension of any structure built in ZF to a rigid structure. Then, even the relational structure, being considered in order to say that the objects are relational discernible, can be extended to a rigid one encompassing monadic predicates as well. (Quine again: [1986], pp. 25-6—if Fa, then $\exists x(x = a \wedge Fx)$, then put $Ax =_{\mathsf{def}} x = a$)

A second argument (really, the same one but written differently):

In extensional set theory, $R(a, b)$, this means $\langle a, b\rangle \in R$, that is,

$\{\{a\}, \{a, b\}\} \in R$. So $\{a\} \in \bigcup R$ (the union of R, which can always e formed within ZF).

Since we can always get $\bigcup R$, we arrive at the *property* "being identical with a", which we can define—like Quine, above—as $I_a(x) \leftrightarrow x = a$.

Since this can be done for every individual of the (finite) domain, we get the monadic properties that makes each one of them an individual, for being $a \neq b$, for sure $I_a(a)$ but $\neg I_a(b)$.

So, PII-A holds.

Muller & Saunders [2008] are right in saying that "absolute discernibles are always weak discernibles".

Right! But the converse also holds!

Saunders *relational* objects are also individuals, as he recognizes, but they are also absolute discernible too. And this is due to the used underlying logic —the classical one!

So, if fermions (in finite number) are to obey classical theory of identity, they are able to bear proper names (according to the standard theory of direct reference), be ordered (in the finite case), etc.

Are we really prepared for such consequences with regard quantum entities, and for fermions on general? I doubt that.

What with QM?

If we assume (I):

- (i) that quantum objects come always in finite number (which we may in principle suppose), and
- (ii) they are (already) individuals, elements of sets.
- (iii) that the underlying logic is classical logic (of first or higher order –including set theory)

The we can always extend the quantum mechanical language (to language with the identity predicates for every quantum object. This makes then individuals in the traditional sense (as classical particles).

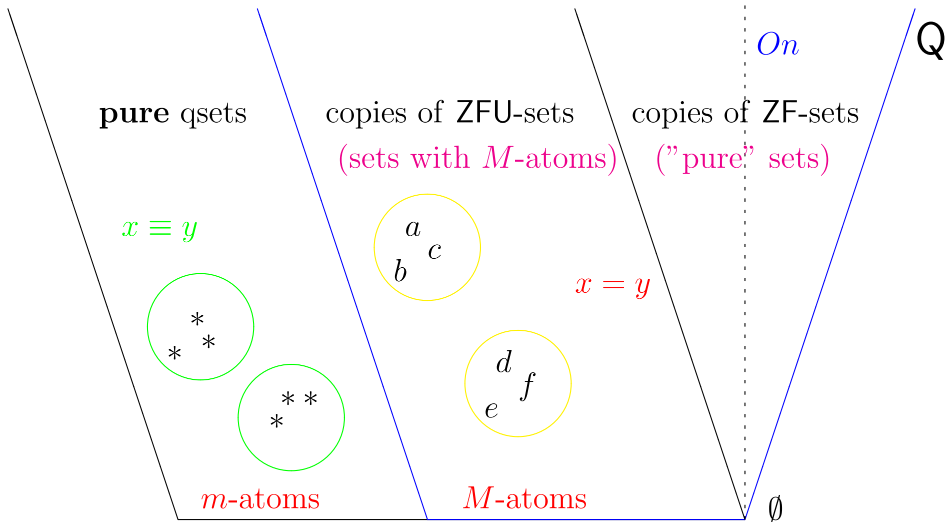
Perhaps we can agree with Muller & Saunders in that "[Q]uantum mechanical description of composite systems of an arbitrary number of similar fermions in all their admissible states, mixed or pure, for all finite-dimensional Hilbert spaces, is *not* in conflict with PII."

But what the above argumentation show is that *nothing* described by classical logic and mathematics can conflict PII !

Okay, you may say that we would avoid the consideration ofthe "problematic" property I_a above and consider just a weaker relation of indiscernibility.

In this case, you are in the direction of quasi-set theory (French & Krause [2006], chap.7).

The quasi-set universe



$$m = \text{"}m\text{-atoms"}; \; M = \text{"}M\text{-atoms"}$$

$$A = m \cup M$$

$$V_0 = A$$

$$V_1 = V_0 \cup \mathcal{P}(V_0)$$

$$\vdots$$

$$V_{n+1} = V_n \cup \mathcal{P}(V_n)$$

$$\vdots$$

$$V_\lambda = \bigcup_{\beta < \lambda} V_\beta, \text{ for } \lambda \text{ a limit ordinal,}$$

$$V = \bigcup_{\alpha \in On} V_\alpha$$

The theory does not "destroy" neither classical logic nor standard mathematics,

But enlarge their scope, enabling us to talk of collections (quasi-sets) of really indiscernible objects, which (by the postulates of the theory) may have a cardinal, but not an associated ordinal.

In the "classical" part of the theory, we can continue to do all the things we do in standard ZF, inclusive to talk about identity and diversity of certain objects.

Thesis: If quantum objects (particles, fields, whatever you mean by quantum objects) are to be indistinguishable, they should be considered as so from the beginning, and not *made* indiscernible by some mathematical device, like the restriction of properties/relations as we do when we opt for being restricted to the walls of some structure.

> "I (...) am persuaded that common speech is full of vagueness and inaccuracy (...) For technical purposes [like *some* discussions in philosophy], technical languages differing [or enlarging] of those of daily life [and of classical logic] are [may be] indispensable." (Russell [1957])

References

French, S. and Krause, D. [2006]: *Identity in Physics: A Historical, Philosophical, and Formal Analysis*, Oxford: Oxford Un. Press.

Domenech, G., Holik, F. and Krause, D. [2008]: 'Q-spaces and the foundations of quantum mechanics', *Foundations of Physics* (On Line). [arXiv:0803.4517v1 [quant-ph]]

THANK YOU