Quasi-set theory: basic ideas

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1 The quasi-set universe

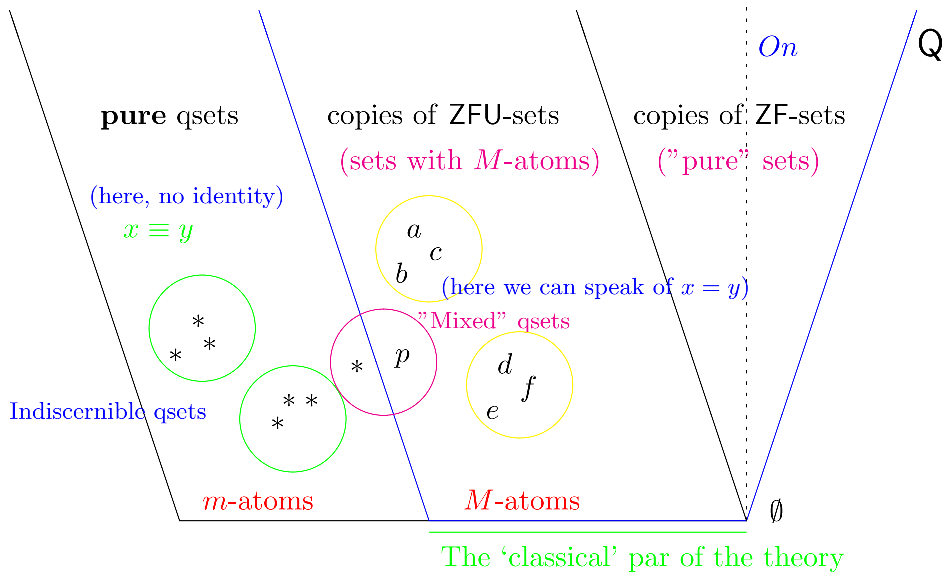

Figure 1: The Quasi-Set Universe: On is the class of ordinals, defined in the 'classical' part of the theory. This 'classical part' contains a copy of ZFU— and of ZF. Those qsets whose transitive closure doesn't contain m-atoms are the "sets" of $\mathfrak{Q}$; they obey the predicate Z.

2 The quasi-set theory $\mathfrak{Q}$

2.1 The language of the formal theory

(i) propositional connectives,

(ii) quantifiers

(iii) individual variables,

(iv) two binary predicates $\equiv$ and $\in$,

(v) three unary predicates m, M and Z, and

(vi) an unary functional symbol qc.

Definition 1

(i) $Q(x) =_{\mathsf{def}} \neg(m(x) \vee M(x))$ (x is a qset)

(ii) $P(x) =_{\mathsf{def}} Q(x) \wedge \forall y(y \in x \to m(y)) \wedge \forall y \forall z(y \in x \wedge z \in x \to y \equiv z)$
x is a pure qset, having m-atoms as elements, not necessarily indiscernible from one each other.

(iii) $D(x) =_{\mathsf{def}} M(x) \vee Z(x)$
x is a *Ding*, a "classical object" in the sense of Zermelo's set theory, namely, either a set or a macro *Urelemente*.

(iv) $E(x) =_{\mathsf{def}} Q(x) \wedge \forall y(y \in x \to Q(y))$
x is a qset whose elements are qsets.

(v) $x =_E y =_{\mathsf{def}} (Q(x) \wedge Q(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge \forall_Q z(x \in z \leftrightarrow y \in z))$
Extensional identity)—we shall write simply $x = y$ instead of $x =_E y$ from now on.

(vi) $x \subseteq y =_{\mathsf{def}} \forall z(z \in x \to z \in y)$ (subqset)

2.2 The postulates of $\mathfrak{Q}$

($\equiv_1$) $\forall x(x \equiv x)$

($\equiv_2$) $\forall x \forall y(x \equiv y \to y \equiv x)$

($\equiv_3$) $\forall x \forall y \forall z(x \equiv y \wedge y \equiv z \to x \equiv z)$

($=_4$) $\forall x \forall y(x = y \to (\alpha(x) \to \alpha(y)))$

($\in_1$) $\forall x \forall y(x \in y \to Q(y))$
If something has an element, then it is a qset; in other words, the atoms have no elements (in terms of the membership relation).

($\in_2$) $\forall_D x \forall_D y(x \equiv y \to x = y)$
Indistinguishable *Dinge* are extensionally identical. This makes $=$ and $\equiv$ coincide for this kind of entities.

($\in_3$) $\forall x \forall y[(m(x) \wedge x \equiv y \to m(y)) \wedge (M(x) \wedge x = y \to M(y)) \wedge (Z(x) \wedge x = y \to Z(y))]$

($\in_4$) $\exists x \forall y(\neg x \in y)$
This qset will be proved to be a set (in the sense of obeying the predicate Z), and it is unique, as it follows from extensionality. Thus, from now own we shall denote it, as usual, by '$\emptyset$'.

($\in_5$) $\forall_Q x(\forall y(y \in x \to D(y)) \leftrightarrow Z(x))$
This postulate grants that something is a set (obeys Z) iff its transitive closure does not contain m-atoms. That is, *sets* in $\mathfrak{Q}$ are those entities obtained in the 'classical' part of the theory (see figure 1 once more).

($\in_6$) $\forall x \forall y \exists_Q z(x \in z \wedge y \in z)$

($\in_7$) If $\alpha(x)$ is a formula in which x appears free, then

$$\forall_Q z \exists_Q y \forall x(x \in y \leftrightarrow x \in z \wedge \alpha(x)).$$

This is the Separation Schema; notation

$$[x \in z : \alpha(x)].$$

When this qset is a set, we write, as usual, $\{x \in z : \alpha(x)\}$.

($\in_8$) $\forall_Q x(E(x) \to \exists_Q y(\forall z(z \in y \leftrightarrow \exists w(z \in w \wedge w \in x)))$.
The union of x, writen $\cup x$. Usual notation is used in particular cases.

2.3 Some basic concepts

From $(\in_6)$: $\forall x \forall y \exists_Q z (x \in z \wedge y \in z)$

Using $\alpha(w) \leftrightarrow w \equiv x \vee w \equiv y$, we get a subqset of z which we denote

$$[x, y]_z$$

which is the qset of the indiscernibles of either x or y that belong to z.

When $x \equiv y$, this qset reduces to

$$[x]_z$$

called the qset of the indiscernble from x that belong to z.

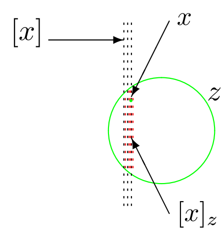

Figure 2: The qset z and one of its elements, x. The collections $[x]$ and $[x]_z$ stand for *all* indiscernible from x and the qset of the indiscernible from x that belong to z respectively. (Usually, $[x]$ is too big to be a qset.)

Later, with the postulates of quasi-cardinal, we will be able to prove $[x]_z$ has a subqset with quasi-cardinal equals to 1:

$$[[x]]_z$$

We call it the strong singleton of x (really, *a* strong singleton, for we cannot grant that it is unique). It has just one element, and we can think that this element *as if* it is x.

Really, $[[x]]_z$ contains *an object of the kind x.*

2.4 More postulates and definitions

($\in_9$) $\forall_Q x \exists_Q y \forall z (z \in y \leftrightarrow w \subseteq x)$,
The power qset of x, denoted $\mathcal{P}(x)$.

($\in_{10}$) $\forall_Q x (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup [y]_x \in x))$,
The infinity axiom.

($\in_{11}$) $\forall_Q x (E(x) \wedge x \neq \emptyset \rightarrow \exists_Q y (y \in x \wedge y \cap x = \emptyset))$,
The axiom of foundation, where $x \cap y$ is defined as usual.

Definition 2 (Weak ordered pair)

$$\langle x, y \rangle_z =_{\mathsf{def}} [[x]_z, [x, y]_z]_z \tag{1}$$

Then, $\langle x, y \rangle_z$ takes all indiscernible from either x or y that belong to z, and it is called the "weak" ordered pair, for it may have more than two elements. Sometimes the sub-indice z will be left implicit.

Definition 3 (Cartesian Product) *Let z and w be two qsets. We define the cartesian product $z \times w$ as follows:*

$$z \times w =_{\mathsf{def}} [\langle x, y \rangle_{z \cup w} : x \in z \wedge y \in w] \tag{2}$$

Functions and relations cannot also be defined as usual, for when there are m-atoms involved, a mapping may not distinguish between arguments and values. Thus we provide a wider definition for both concepts, which reduce to the standard ones when restricted to classical entities. Thus,

Definition 4 (Quasi-relation) *A qset R is a binary quasi-relation between to qsets z and w if its elements are weak ordered pairs of the form $\langle x, y \rangle_{z \cup w}$, with $x \in z$ and $y \in w$.*

Definition 5 (Quasi-functions) *Quasi-functions: injective, sobrejective, bijective.*

2.5 Postulates for quasi-cardinals

Here α, β, ... stand for cardinals (defined in the classical part of the theory):

(qc_1) $\forall_Q x(\exists_Z y(y = qc(x)) \to \exists! y(Cd(y) \wedge y = qc(x) \wedge (Z(x) \to y = card(x)))$
If the qset x has a quasi-cardinal, then its (unique) quasi-cardinal is a cardinal (defined in the 'classical' part of the theory) and coincides with the cardinal of x stricto sensu if x is a set.

(qc_2) $\forall_Q x(x \neq \emptyset \to qc(x) \neq 0)$.
Every non-empty qset has a non-null quasi-cardinal.

(qc_3) $\forall_Q x(\exists_Z \alpha(\alpha = qc(x)) \to \forall \beta(\beta \leq \alpha \to \exists_Q z(z \subseteq x \wedge qc(z) = \beta)))$
If x has quasi-cardinal α, then for any cardinal $\beta \leq \alpha$, there is a sub-qset of x with that quasi-cardinal.

In the remaining axioms, for simplicity, we shall write $\forall_{Q_{qc}} x$ (or $\exists_{Q_{qc}} x$) for quantifications over qsets x having a quasi-cardinal.

(qc_4) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(y \subseteq x \to qc(y) \leq qc(x))$

(qc_5) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(Fin(x) \wedge x \subset y \to qc(x) < qc(y))$

It can be proven that if both x and y have a quasi-cardinal, then $x \cup y$ has a quasi-cardinal. Then,

(qc_5) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(\forall w(w \notin x \vee w \notin y) \to qc(x \cup y) = qc(x) + qc(y))$

In the next axiom, $2^{qc(x)}$ denotes (intuitively) the quantity of subquasi-sets of x. Then,

(qc_6) $\forall_{Q_{qc}} x(qc(\mathcal{P}(x)) = 2^{qc(x)})$

2.6 The Weak Extensionality Axiom

($\equiv_5$) $\forall_Q x \forall_Q y((\forall z(z \in x/\equiv \to \exists t(t \in y/\equiv \wedge \wedge QSim(z,t)))) \wedge \forall t(t \in y/\equiv \to \exists z(z \in x/\equiv \wedge \wedge QSim(t,z))) \to x \equiv y)$

The following theorem express the invariance by permutations in $\mathfrak{Q}$, and with this result we finish our revision:

Theorem 1 *[Invariance by Permutations] Let x be a finite qset such that $\neg(x = [z]_t)$ for some t and let z be an m-atom such that $z \in x$. If $w \in t$, $w \equiv z$ and $w \notin x$, then there exists $[[w]]_t$ such that*

$$(x - [[z]]_t) \cup [[w]]_t \equiv x$$

This above theorem is illustrated by the figure 3 below, where $[z]$ is the collection—or "quasi-class"—of all indiscernibles of z, while $[z]_t$ is given by the pair axiom and the separation schema:

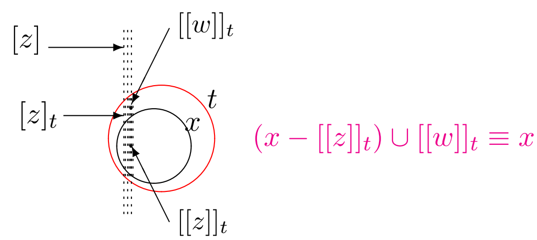

Figure 3: The invariance by permutations in $\mathfrak{Q}$. Two indiscernible elements from $z \in x$ and $w \notin x$, expressed by their quasi-singletons $[[z]]_t$ and $[[w]]_t$, are "interchanged" and the resulting qset x remains indiscernible from the original one. The hypothesis that $\neg(x = [z]_t)$ grants that there are indiscernible from z in t which do not belong to x.

3 Relative Consistency

It is easy to see (as put by French & Krause) that there is a translation from the language of ZFU ($\mathcal{L}_{ZFU}$) to the language of $\mathfrak{Q}$, so that if $\mathfrak{Q}$ is consistent, so is ZFU (and hence so is ZF). The translation can be described as follows, if we suppose that $\mathcal{L}_{ZFU}$ has a primitive unary symbol S so that $S(x)$ says intuitively that x is a set. Then, being A any formula of the language of ZFU, let A^q be its translation in the language of $\mathfrak{Q}$, defined as follows:

Definition 6

(a) If A is $S(x)$, then A^q is $Z(x)$

(b) If A is $x = y$, then A^q is $((M(x) \wedge M(y)) \vee (Z(y) \wedge Z(y)) \wedge x =_E y)$

(c) If A is $x \in y$, then A^q is $((M(x) \vee Z(x)) \wedge Z(y)) \wedge x \in y$

(d) If A is $\neg B$, then A^q is $\neg B^q$

(e) If A is $B \vee C$, then A^q is $B^q \vee C^q$

(f) If A is $\forall x B$, then A^q is $\forall x(M(x) \vee Z(x) \rightarrow B)$

(g) Finally, the term $qc(x)$ is interpreted in $card(x)$.

It is immediate that $\mathsf{Cons}(\mathfrak{Q}) \mapsto \mathsf{Cons}(\mathrm{ZFU})$, where the predicate Cons stands for "consistency of". Figure 1 above shows the intuitive aspects of this definition.

Now let us consider in brief the converse of this result.

In 'pure' ZF (without *Urelemente*):

(1) $m \neq \emptyset$ a non empty set

(2) R is an equivalence relation on m.
The equivalence classes of the quotient set m/R are denoted by $C_1, C_2 \ldots$.

(3) If $x \in m$, define

$$\widehat{x} = \langle x, C_x \rangle,$$

where C_x is the equivalence class to which x belongs and call $\widehat{m}$ the set of all $\widehat{x}$ with $x \in m$.

The basis:
$X = \widehat{m} \cup M$,
where $\widehat{m}$ is as above and M is a set such that $\widehat{m} \cap M = \emptyset$ and
$rank(\widehat{m}) = rank(M)$.

Then we define a superstructure $\mathbf{Q}$ over the set X, called the *Q-set universe* (see Figure 4). As we will see, $\mathbf{Q}$ acts as a 'model' for the quasi-set theory $\mathcal{Q}$. The definition is as follows:

$$Q_0 =_{\mathsf{def}} X$$
$$Q_1 =_{\mathsf{def}} X \cup \mathcal{P}(X)$$
$$\vdots$$
$$Q_\lambda =_{\mathsf{def}} \bigcup_{\beta<\lambda} Q_\beta \qquad \text{if } \lambda \text{ is a limit ordinal}$$
$$\mathbf{Q} =_{\mathsf{def}} \bigcup_{\alpha\in On} Q_\alpha.$$

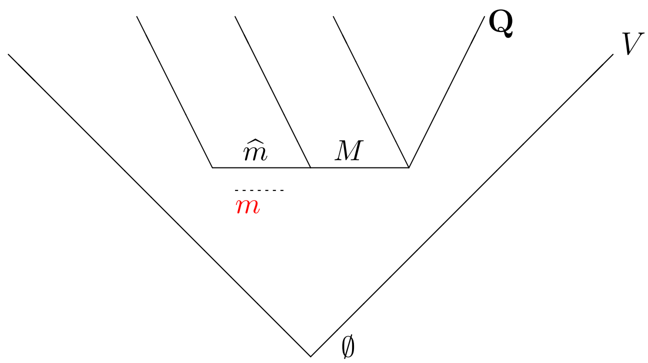

Figure 4: Simulating qsets in V (the ZFC-universe). The elements of m—dashed lines—are outside $\mathbf{Q}$. Thus, *within* the structure $\mathbf{Q}$, we cannot "see" that the elements of m are individuals.

The structure $\mathbf{Q}$ seems to be not rigid.